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On the Application of Harmonic Analysis to the Dynamical Theory of the Tides. Part I. On Laplace's "Oscillations of the First Species," and on the Dynamics of Ocean Currents

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IX. *On the Application of Harmonic Analysis to the Dynamical Theory of the Tides.*—Part I. *On* LAPLACE'S *"Oscillations of the First Species," and on the Dynamics of Ocean Currents.*

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Communicated by Professor G. H. DARWIN, *F.R.S.*

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THE earliest attempt to subject the Theory of the Tides to a rigorous dynamical treatment was given by LAPLACE in the first and fourth books of the 'Mécanique Céleste.' The subject has since been treated by AIRY,* KELVIN,† DARWIN,‡ LAMB,§ and other writers, but with the exception of the extension of LAPLACE'S results to include the theory of the long-period tides, but little practical advance has been made with the subject, in spite of the enormous increase in the power of the mathematical resources at our disposal, and the problem has remained in very much the same condition as it was left by LAPLACE. This arises no doubt partly from the difficulties inherent to the subject, but partly from the form in which the theory was originally presented by LAPLACE in the 'Mécanique Céleste,' which has been described by AIRY as "perhaps on the whole more obscure than any other part of the same extent in that work." The obscurity complained of does not however seem to have been entirely removed by LAPLACE'S successors, and it was the fact that every presentment of the theory with which I was acquainted offered some points of difficulty, that in the first instance led me to take up the problem *ab initio*, partly with the purpose of allaying the doubts which had arisen in my own mind as to the validity of certain approximations employed by LAPLACE and adopted by his successors, and partly in the hope that I might be able to extend the results of LAPLACE to meet more fully the case presented by the circumstances actually existent in Nature.

Up to the present I have been unable to free the problem to any extent from the limitations which have been imposed by previous writers, and consequently it would be futile to claim that the results I am now able to put forward materially advance

* 'Encyc. Metropolitana'; Art., "Tides and Waves," Section III.

† 'Phil. Mag.,' 1875, vol. 50.

‡ 'Encyc. Britannica' (9th edition); Art., "Tides."

§ 'Hydrodynamics,' chapter viii.

our knowledge of the tides as they actually exist; but I venture to hope that these results, as applied to the oscillations of an ideal ocean, considerably simpler in character than the actual ocean, may prove of some interest from the point of view of pure hydrodynamical theory.

In §§ 1–4 I have devoted considerable space to the formation of the dynamical equations. The equations obtained agree with those used by LAPLACE, and consequently it may be thought that I have been unnecessarily diffuse over this part of the subject. My apology is that the questionable, if not erroneous, reasoning which has often been assigned for the various approximations introduced seemed to me to warrant a very minute examination of the formation of these equations. An analytical treatment, such as that I have used, seems to me to be the only safe method of procedure to ensure that the approximations do not involve the neglect of any terms which may be of equal importance with those retained, many of which are extremely small. The method adopted follows LAPLACE in so far as it consists of a transformation of the general equations of oscillation of a rotating fluid. I trust however that these general equations in the form I have used, which seems to be the simplest form to which they can be reduced, may be found less "repulsive" than those employed by LAPLACE.

§ 5 deals with the integration of these equations. The forms of solution discussed in the present paper are those which are symmetrical with respect to the axis of rotation. The types of oscillation represented by these solutions have been named by LAPLACE, "Oscillations of the First Species," but he omitted to discuss them in detail on the grounds that the oscillations of such character, which might be expected to exist in Nature, would be modified to such an extent by friction that they would be far better represented by the old "equilibrium theory," than by a dynamical theory which failed to take due account of the action of friction. The tides in question will be of long period, the shortest of the periods being half a lunar month in duration, but Professor DARWIN was, I believe, the first to call attention to the fact that this length of period will hardly be sufficiently great to render the effects of friction of such paramount importance, and hence he added to the work of LAPLACE a discussion of the long-period tides when not subject to frictional influences. I have recently attempted to estimate the effects of friction on the tidal oscillations of the ocean,* and the results at which I have arrived fully confirm the view of Professor DARWIN as to the small influence of friction on the lunar-fortnightly tides, and render it highly probable that the effects will be almost equally slight on the solar long-period tides.

The method of integration I have employed differs from that used by DARWIN, my aim having been to express the results by means of series of zonal harmonics instead of by the power-series obtained by him. The advantages of this are two-fold; firstly,

* In a paper read before the London Math. Soc., December 10th, 1896.

it allows of our including in our analysis the effects due to the gravitational attraction of the water; and secondly, the convergence of the series obtained will be much more rapid. The latter circumstance is of particular value, as it has enabled me to treat with considerable success the problem of the free oscillations of the ocean.

The general problem of the small oscillations of a rotating system possessing a finite number of degrees of freedom has been discussed by THOMSON and TAIT;* but the extension to meet the case where the number of degrees of freedom is infinite involves analytical considerations of some delicacy. As a rule, the transition from the case of a system with finite freedom to that of a system with infinite freedom is effected by the employment of "normal coordinates,"† and the chief difficulty in the solution of problems relating to the vibrations of the latter class of system consists in the discovery of these coordinates. The researches of THOMSON and TAIT just mentioned shew however that in a rotating system these normal coordinates do not exist, and hence that the methods ordinarily employed to deal with the oscillations of a system about a state of equilibrium will no longer suffice for the treatment of our problem. In most "gyrostatic" problems which have been solved hitherto,‡ the solution has been obtained by means of a system of quasi-normal coordinates. When such coordinates exist, only a finite number of oscillations of certain particular types are possible, and, by constraining the system to vibrate in one of these types, we may treat it in the same manner as a system with a finite number of degrees of freedom. The period-equation for the free oscillations of an assumed type will then only possess a finite number of roots, and will consequently be an algebraic equation usually most readily obtained in a determinantal form. It is shewn at the end of § 5 that the coordinates we have used possess this property when the depth of the ocean follows certain restricted laws; but in general no such quasi-normal coordinates exist, and whatever coordinates be employed, the displacements in any of the fundamental modes of vibration can only be expressed by means of an infinite number of such coordinates. The most advantageous choice of coordinates will then be that which leads to most rapidly converging series.

As however the oscillations of an assumed type can only be expressed by an infinite series of coordinates, it follows that an infinite number of oscillations of any assumed type must be possible, and that consequently the period-equation for oscillations of this type will have an infinite number of roots and will therefore be transcendental instead of algebraic in character. It is possible that the transition from systems with finite freedom to systems with infinite freedom may be treated with advantage by the employment of determinants of infinite order (as a means of expressing the transcendental period-equation), after the manner introduced into analysis by

* 'Natural Philosophy,' Part I., § 345.

† RAYLEIGH, 'Theory of Sound,' vol. 1, § 87.

‡ *Cf.* POINCARÉ, 'Acta Mathematica,' vol. 7; BRYAN, 'Phil. Trans.,' 1889.

G. W. HILL, in his 'Researches on the Lunar Theory;'* but in the present instance we are able to avoid the difficulties involved in the use of these infinite determinants, in that the forms of determinant which occur are those which are associated with continued fractions.

§ 6 deals with the analytical discussion of the deduction of the period-equation. The method is based on a paper by Lord KELVIN,† in which the author defends the procedure of LAPLACE against certain allegations to which it had been subjected by AIRY, but I have endeavoured to present the arguments in a somewhat different light, so as to bring out more clearly the analogy between our problem and the general problem of vibrating systems with finite freedom.

In §§ 7–10 I have given illustrations of the method of solving the period-equation numerically, and of the subsequent determination of the type of motion for the different fundamental modes. As the ground covered in these sections is almost entirely new, I have devoted considerable time and labour to the arithmetical determination of the periods and types of the principal oscillations for a system comparable with the earth in magnitude. The results are tabulated in these sections.

§ 11 deals briefly with the forced tides of long period due to the moon in an ocean of uniform depth. The results agree with those previously obtained by other methods, but differ from them in analytical form. In § 12 I have given illustrations of a means of extending the method of numerical computation to cases where the law of depth is less restricted in character.

The consideration of forced tides of very long period, dealt with in § 13, points to the existence of free oscillations of infinitely long period. This, I believe, was first noted by Professor LAMB,‡ but the application of the dynamical equations for the tides to the treatment of these free oscillations has not been previously carried out. The types of motion in question appear to be of considerable importance, as they throw light on a phenomenon which in the past has been the subject of considerable controversy. The difficulties which have been met with in attempts to account for the existence of ocean currents all seem to me to arise from an over-estimate of the effects of viscosity on the motion of the sea. The large-scale ocean currents have been attributed by Sir JOHN HERSCHEL§ and others entirely to the influence of the "trade" and other prevailing winds, which give rise to slow steady motions which, in the absence of friction, would remain permanent even were the originating cause entirely to cease. The difficulty in accepting this view arises from the assumption that such currents would succumb to the influence of frictional

* 'Acta Mathematica,' vol. 8.

† 'Phil. Mag.,' 1875, p. 227.

‡ 'Hydrodynamics,' § 198.

§ 'Physical Geography,' §§ 57–60.

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resistances in the course of a few days.* If this assumption be correct it will of course be necessary to invoke some more constant cause than the fickle winds in explanation of ocean currents, but unfortunately the causes put forward by the chief opponents of the wind theory, namely, the differences of density arising from differences of temperature, salinity, &c., though no doubt satisfying the criterion of being more constant in their action, seem to be equally ineffective in maintaining the currents against such large resistances as would be required to destroy the currents due to the winds in a few days. If, on the other hand, the period of subsidence of the free current-motions is to be reckoned rather by years, these motions could not fail to be excited and maintained by such causes as the winds even against the action of friction.

In § 14 I have dealt with the dynamics of ocean currents on the supposition that they are of the nature of free steady motions (probably maintained by a variety of causes), and that the influences of viscosity are extremely small. A remarkable result is the extremely restricted character of the possible forms of steady motion as contrasted with the case where the ocean covers a non-rotating globe, in which latter case the possible forms of steady motion are to a large extent arbitrary. It is found that if the density of the water is uniform, the only forms of steady motion possible when the depth depends on the latitude alone are those in which the water always moves along parallels of latitude, while in general the paths of the fluid particles coincide with certain lines depending only on the distribution of land and water and on the configuration of the ocean bed. The equation by which these lines are defined is of an extremely simple character, and from it we could at once trace out the forms of the stream-lines on a chart if we had a sufficient knowledge of the configuration of the ocean bed. The equator will always be one of these stream-lines, and herein we seem to have the explanation of the fact that the ocean-currents always tend to set along the equator, but in other respects it is shewn that the effects of variations of density will seriously interfere with the simple laws which must hold so long as the density is uniform.

The importance of the earth's rotation in influencing ocean-currents has long been recognised by physicists, but I am not aware that any previous attempt has been made to investigate this influence mathematically. The numerical results obtained in § 15 are interesting, as showing how a cause, which on a non-rotating globe could not give rise to any appreciable currents, may be rendered highly effective in maintaining currents as a consequence of the rotation of the earth.

In attempting to account for ocean currents, the real question at issue is: How far are the suggested causes capable of maintaining currents *against the action of friction?* To answer this question an investigation, either mathematical or experimental, as to the effects of friction is essential. Such an investigation I have endeavoured

* MAURY, 'Physical Geography of the Sea,' § 93.

to supply in another paper,* but mathematical difficulties have compelled me, in treating of friction, to omit from consideration the important influences due to the rotation. We have already called attention to the fact that the free steady motions on a non-rotating globe are far less restricted in character than those on a rotating globe, while, in that the latter essentially violate what appears to be a necessary condition when the water is viscous, namely, that there can be no slipping at the bottom, it seems to me to be probable that even the limited forms of steady motion here dealt with would be no longer possible if the water were viscous, but that, if they were started by any means, they would at once give place to periodic motions of comparatively short period.† This conclusion has been forced on me by the apparent impossibility of satisfying the equations of motion of a viscous ocean on a rotating globe by means of slowly declining current-motions. If such should be the case, it follows that no stable currents can exist without variations in the density of the water. As however I have not as yet been able to support this view by anything approaching a rigorous mathematical treatment, the question must for the present remain open.

§ 1. *Differential Equations for the Vibration of a Rotating Mass of Liquid.*

Suppose we are dealing with the small oscillations of a mass of liquid about a state of steady motion consisting of a rotation as a rigid body with angular velocity ω about a certain axis.

Take this axis as axis of z, and refer to a set of rectangular axes rotating about it with uniform angular velocity ω. Then, in the steady motion supposed, the fluid will have no motion relatively to these axes.

Let u, v, w denote the relative velocity-components at the point x, y, z due to the small oscillations. The actual velocity-components parallel to the instantaneous positions of the moving axes will then be

$$u - \omega y, \quad v + \omega x, \quad w,$$

and, therefore, if we suppose the amplitude of the vibrations sufficiently small to allow of our neglecting squares and products of the small quantities u, v, w, the differential equations of motion of the liquid may be written in the form‡

* *Loc. cit., ante.*

† The condition that there can be no slipping at the bottom will reduce the number of degrees of freedom of the system, and hence we may anticipate that certain types of motion which were possible before this condition was imposed will no longer exist afterwards.

‡ BASSET, 'Hydrodynamics,' vol. 1, p. 22.

$$\frac{\partial u}{\partial t} - \omega (v + \omega x) - \omega v = \frac{\partial}{\partial x} (V' - p/\rho),$$
$$\frac{\partial v}{\partial t} + \omega (u - \omega y) + \omega u = \frac{\partial}{\partial y} (V' - p/\rho),$$
$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial z} (V' - p/\rho),$$

where V' denotes the potential of the bodily forces acting on the liquid, p the fluid pressure, and ρ the density.

If now we put

$$\psi = V' - p/\rho + \tfrac{1}{2}\omega^2 (x^2 + y^2) + \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

the above equations reduce to

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\omega v &= \frac{\partial \psi}{\partial x} \\ \frac{\partial v}{\partial t} + 2\omega u &= \frac{\partial \psi}{\partial y} \\ \frac{\partial w}{\partial t} &= \frac{\partial \psi}{\partial z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

while the incompressibility of the liquid is expressed by the additional equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

These equations, originally given by POINCARÉ,* suffice, in conjunction with certain conditions which must hold at the boundary, for the determination of the four functions u, v, w, ψ. They are perhaps the simplest equations of which to make use when dealing with the oscillations of a mass of liquid of finite extent in three dimensions, and, for this purpose, they were first solved by POINCARÉ in a form adapted for satisfying boundary-conditions at an ellipsoidal surface, while additional applications have been considered by BRYAN,† LOVE,‡ and myself.§ The possibility of solution in each of these cases however turns on the fact that it only required to satisfy boundary-conditions at a single ellipsoidal or spheroidal surface, whereas, in the problem presented by the tides, it is necessary to satisfy conditions at two surfaces, namely, the ocean bed and the free surface of the ocean.

There is, however, a feature attached to this problem which enables us to surmount

* 'Acta Mathematica,' vol. 7, p. 356.
† 'Phil. Trans.,' 1889.
‡ 'Proc. Lon. Math. Soc.,' vol. 19.
§ 'Phil. Trans.,' 1895.

the difficulties arising from this cause; the fluid which constitutes the ocean may be regarded as a thin layer distributed over an approximately spherical surface, a circumstance which enables us to reduce the number of our independent variables and to treat the problem as a two-dimensional one.

Before proceeding to the transformation of our equations, let us examine them in the form in which they are given above. If we suppose the system is executing a simple harmonic vibration in period $2\pi/\lambda$, we may put u, v, w, ψ each proportional to $e^{i\lambda t}$, and therefore replace $\partial u/\partial t$, &c., by $i\lambda u$, &c. Thus the equations (2) give

$$i\lambda u - 2\omega v = \frac{\partial\psi}{\partial x},$$
$$i\lambda v + 2\omega u = \frac{\partial\psi}{\partial y},$$
$$i\lambda w \quad = \frac{\partial\psi}{\partial z}.$$

Now in an important class of oscillations, viz., the tides of long period, the value of λ will be small compared with that of ω; while for another class of motions, viz., the steady ocean-currents, we must suppose λ absolutely zero. In these cases, if we retain only the most important terms, the equations of motion take the approximate form

$$-2\omega v = \frac{\partial\psi}{\partial x}, \qquad 2\omega u = \frac{\partial\psi}{\partial y}, \qquad 0 = \frac{\partial\psi}{\partial z}.$$

Hence, applying the operators $\partial/\partial z$ to the first two, and making use of the third, we find

$$\partial u/\partial z = 0, \qquad \partial v/\partial z = 0.$$

Likewise also from the equation of continuity,

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -\frac{1}{2\omega}\left(\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial^2\psi}{\partial y\,\partial x}\right) = 0.$$

From this we see that in the case of tides of very long period the velocity of the fluid particles is approximately the same at all points in the same line parallel to the polar axis, while in the case of the ocean-currents this is rigorously the case.

Now in order to effect the transformation of the equations of motion, it has been assumed by LAPLACE and his followers that, on the analogy of "long waves" when there is no rotation, all fluid particles which are at one instant in a vertical line will remain in such a line. This assumption appears to require some modification in the case of our rotating system. We shall see hereafter however that the assumption in question will not lead to appreciable error, provided that the depth of the water is small in comparison with the radius of the solid globe on which it resides, a hypothesis which will certainly be applicable in the case of the earth.

§ 2. *The Boundary-conditions.*

Before proceeding with the approximations which we propose to employ hereafter, let us examine the boundary-conditions to which the functions u, v, w, ψ are subject in the general case.

Suppose the fluid resides on the surface of a solid nucleus which is constrained to rotate with uniform angular velocity ω about the axis of z. We introduce this constraint so as to avoid the complications resulting from the reactions of the fluid motion on that of the nucleus. Since in the case of the earth the mass of the ocean is exceedingly small compared with that of the solid parts, such reactions would be very minute, while for most of the more important types of oscillation they would not exist at all. In such types the problem is not affected by the introduction of the constraints. The boundaries of the ocean where it is in contact with the solid nucleus may then be regarded as fixed relatively to the moving axes Ox, Oy, Oz, and the condition to be satisfied at these boundaries is that there is no flow of fluid across them. Denoting by l, m, n the direction-cosines of the normal to the surface, this condition is expressed analytically by the equation

$$[lu + mv + nw] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Next, let us examine the boundary-conditions at the free surface. Let l, m, n denote the direction-cosines of the normal to this surface in its undisturbed position, and let ζ be the distance between the displaced surface and the mean surface measured along the normal to the latter. Then we may equate the velocity at the mean surface in the direction of this normal to the rate at which ζ increases; thus at the undisturbed surface we have

$$[lu + mv + nw] = \partial\zeta/\partial t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Lastly, we must express the condition that the pressure at the actual free surface is zero (or constant). Now if dn' denote an element of the normal to the undisturbed surface, and $\overline{p}$, $\overline{\frac{\partial p}{\partial n'}}$ denote the values at this surface of the pressure and its rate of increase along the normal, the pressure at the actual surface is

$$\overline{p} + \zeta\,\overline{\frac{\partial p}{\partial n'}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

and this we have seen is to be equated to a constant.

But by definition of ψ we have

$$p/\rho = \text{const} + V' + \tfrac{1}{2}\,\omega^2\,(x^2 + y^2) - \psi$$
$$= \text{const} + V'_0 + v' + v + \tfrac{1}{2}\,\omega^2\,(x^2 + y^2) - \psi \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where V'_0 denotes the potential at x, y, z in the steady motion, v' the potential due to the attraction of the layer of fluid contained between the actual free surface and its mean position, and v the disturbing potential, which may be regarded as due to some external attracting system.

Since in (6) $\overline{\partial p}/\partial n'$ is already associated with the small factor ζ, in calculating $\overline{\partial p}/\partial n'$ we may omit all small quantities of the order ζ, and thus replace this expression by its value in the steady motion. But from (7) we have in this case

$$p/\rho = \text{const} + V'_0 + \tfrac{1}{2}\omega^2\,(x^2 + y^2),$$

whence

$$\frac{1}{\rho}\,\frac{\overline{\partial p}}{\partial n'} = \frac{\partial}{\partial n'}\,\{V'_0 + \tfrac{1}{2}\omega^2\,(x^2 + y^2)\}.$$

Now, since the free surface of the ocean must be an equipotential surface, the resultant of gravitation, including centrifugal force, must be perpendicular to this surface. Denoting by g its value, we have

$$g = -\,\frac{\partial}{\partial n'}\,\{V'_0 + \tfrac{1}{2}\omega^2\,(x^2 + y^2)\},$$

and therefore

$$\frac{1}{\rho}\,\frac{\overline{\partial p}}{\partial n'} = -\,g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Introducing the values of $\overline{p}$, $\frac{\overline{\partial p}}{\partial n'}$ from (7), (8) into the expression (6), and equating the latter to a constant, we find

$$[V'_0 + v' + v + \tfrac{1}{2}\omega^2\,(x^2 + y^2) - \psi] - g\zeta = \text{const},$$

or, on equating periodic parts to zero,

$$\overline{\psi} = \overline{v'} - g\zeta + \overline{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

where the bars are used to denote surface-values at the undisturbed free surface.

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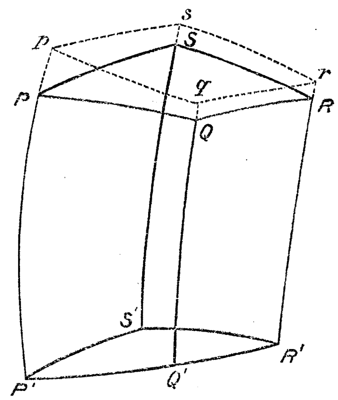
§ 3. *Transformation of the Equation of Continuity.*

We proceed now to the transformation of our equations into a form analogous to that used by LAPLACE, dealing first with the equation of continuity.

Let us refer to a system of orthogonal curvilinear coordinates α, β, γ, and suppose that the undisturbed surface of the ocean coincides with one of the surfaces γ=const, say $\gamma = \gamma_0$.

On the surface $\gamma = \gamma_0$ take a small parallelogram PQRS, bounded by curves of the systems α = const, β = const.



Through the sides of this parallelogram draw the surfaces α = const, β = const, to meet the inner surface of the ocean in the quadrilateral P′Q′R′S′ and the distorted free surface in *pqrs*.

The surfaces α = const, β = const, are, of course, supposed to be in rotation with angular velocity ω, in common with the axes Ox, Oy, Oz. Let U, V, W denote the relative velocity-components at the point (α, β, γ) parallel to the normals to the surfaces of reference and in the directions in which α, β, γ respectively increase, and let $\gamma = \gamma_1$, where γ_1 may be regarded as a function of α, β, be the equation to the surface of the solid earth.

Let h_1, h_2, h_3 be parameters associated with our orthogonal system of coordinates, such that the line-element ds is given by

$$ds^2 = d\alpha^2/h_1^2 + d\beta^2/h_2^2 + d\gamma^2/h_3^2.$$

Then the volume of liquid which flows in a unit of time across the face α = const of an elementary parallelopiped whose adjacent edges are $\delta\alpha/h_1$, $\delta\beta/h_2$, $\delta\gamma/h_3$ is

$$\mathrm{U}\,\frac{\delta\beta}{h_2}\,\frac{\delta\gamma}{h_3}.$$

But if in the above figure we suppose that α, β, γ_0 are the coordinates of P, and that PQ = $\delta\beta/h_2$, PS = $\delta\alpha/h_1$, the total flow of liquid across the face PP′Q′Q will be found by integrating the above expression with respect to γ between the limits $\gamma = \gamma_1$, $\gamma = \gamma_0$.

Thus, the rate at which liquid is entering the elementary volume PR′ across the face PQ′ is expressed by

$$\delta\beta \int_{\gamma_1}^{\gamma_0} (\mathrm{U}/h_2 h_3)\, d\gamma.$$

If in this expression we change α into $\alpha + \delta\alpha$, we shall obtain the rate at which fluid is flowing in the positive direction across the face SR′; therefore, the rate at which fluid is leaving the element across the face SR′ is expressed by

$$\delta\beta \int_{\gamma_1}^{\gamma_0} (\mathrm{U}/h_2 h_3)\, d\gamma + \delta\alpha\, \delta\beta \frac{\partial}{\partial\alpha} \left\{ \int_{\gamma_1}^{\gamma_0} (\mathrm{U}/h_2 h_3)\, d\gamma \right\}.$$

In like manner, the rate at which fluid enters across the face PS′ is

$$\delta\alpha \int_{\gamma_1}^{\gamma_0} (\mathrm{V}/h_3 h_1)\, d\gamma,$$

and that at which it escapes across the face QR′ is

$$\delta\alpha \int_{\gamma_1}^{\gamma_0} (\mathrm{V}/h_3 h_1)\, d\gamma + \delta\alpha\, \delta\beta \frac{\partial}{\partial\beta} \left\{ \int_{\gamma_1}^{\gamma_0} (\mathrm{V}/h_3 h_1)\, d\gamma \right\}.$$

Lastly, in virtue of the boundary-equation (4), which holds at the surface P′Q′R′S′, the rate at which fluid enters over this surface is zero, while in virtue of (5), which holds at the surface PQRS, the rate at which fluid escapes over the latter surface is expressed by

$$\frac{\delta\alpha}{h_1} \frac{\delta\beta}{h_2} \frac{\partial\zeta}{\partial t}.$$

Now the total amount of liquid contained within the elementary volume under consideration is constant, and therefore, if we equate to zero the sum of the rates at which fluid is entering over all the six faces, we obtain the equation of continuity in the form

$$-\delta\alpha\, \delta\beta \frac{\partial}{\partial\alpha} \left\{ \int_{\gamma_1}^{\gamma_0} (\mathrm{U}/h_2 h_3)\, d\gamma \right\} - \delta\alpha\, \delta\beta \frac{\partial}{\partial\beta} \left\{ \int_{\gamma_1}^{\gamma_0} (\mathrm{V}/h_3 h_1)\, d\gamma \right\} - \frac{\delta\alpha}{h_1} \frac{\delta\beta}{h_2} \frac{\partial\zeta}{\partial t} = 0,$$

or

$$\frac{\partial\zeta}{\partial t} = -h_1 h_2 \left[\frac{\partial}{\partial\alpha} \left\{ \int_{\gamma_1}^{\gamma_0} (\mathrm{U}/h_2 h_3)\, d\gamma \right\} + \frac{\partial}{\partial\beta} \left\{ \int_{\gamma_1}^{\gamma_0} (\mathrm{V}/h_3 h_1)\, d\gamma \right\} \right] \quad . \quad . \quad (10).$$

So far no approximation has been made other than that involved in supposing the vibrations small. If we now suppose that the depth of the water is small, so that $\gamma_0 - \gamma_1$ is a small quantity, the above equation admits of considerable simplification. Using square brackets to denote values at the mean surface $\gamma = \gamma_0$, we have, by TAYLOR's Theorem,

$$\mathrm{U}/h_2h_3 = [\mathrm{U}/h_2h_3] + (\gamma - \gamma_0)\left[\frac{\partial}{\partial\gamma}(\mathrm{U}/h_2h_3)\right] + \ldots,$$

whence

$$\int_{\gamma_1}^{\gamma_0}(\mathrm{U}/h_2h_3)\,d\gamma = [\mathrm{U}/h_2h_3](\gamma_0 - \gamma_1) - \tfrac{1}{2}(\gamma_0 - \gamma_1)^2\left[\frac{\partial}{\partial\gamma}(\mathrm{U}/h_2h_3)\right] + \ldots$$

Now, if $\gamma_0 - \gamma_1$ be small as supposed above,* even though $\frac{\partial}{\partial\gamma}(\mathrm{U}/h_2h_3)$ is finite, we may omit all the terms on the right except the first. This amounts to supposing that the horizontal velocity is sensibly uniform throughout the depth, not on account of the small value of its rate of variation, but on account of the small distance through which this variation can take effect, a supposition which is not inconsistent with the results of § 1. Hence, on neglecting small terms of the order $(\gamma_0 - \gamma_1)^2$, we have

$$\int_{\gamma_1}^{\gamma_0}(\mathrm{U}/h_2h_3)\,d\gamma = [\mathrm{U}/h_2h_3](\gamma_0 - \gamma_1),$$

and, in like manner,

$$\int_{\gamma_1}^{\gamma_0}(\mathrm{V}/h_2h_3)\,d\gamma = [\mathrm{V}/h_3h_1](\gamma_0 - \gamma_1).$$

Let h denote the depth of the ocean at the point (α, β). Then, provided h be small in comparison with the radii of curvature of normal sections of the surfaces $\alpha = \text{const}$, $\beta = \text{const}$, $\gamma = \text{const}$, we may put

$$\frac{\gamma_0 - \gamma_1}{[h_3]} = h,$$

with errors of the order of the square of the ratio of h to these radii of curvature; and therefore

$$\int_{\gamma_1}^{\gamma_0}(\mathrm{U}/h_2h_3)\,d\gamma = h\,[\mathrm{U}/h_2],$$

$$\int_{\gamma_1}^{\gamma_0}(\mathrm{V}/h_3h_1)\,d\gamma = h\,[\mathrm{V}/h_1].$$

Substituting these values in (10), we find

$$\frac{\partial\zeta}{\partial t} = -\,\bar{h}_1\bar{h}_2\left[\frac{\partial}{\partial\alpha}\left(\frac{\bar{\mathrm{U}}h}{\bar{h}_2}\right) + \frac{\partial}{\partial\beta}\left(\frac{\bar{\mathrm{V}}h}{\bar{h}_1}\right)\right] \quad \ldots \ldots \ldots \quad (11),$$

where we have now used bars to denote surface values.

* The standard of comparison is considered in the next section.

If we suppose that the free surface of the ocean is a spheroid of revolution about the axis Oz, it will be convenient to refer to a system of spheroidal coordinates μ, ϕ, ν related to x, y, z by the equations

$$x = c\sqrt{(1+\nu^2)}\sqrt{(1-\mu^2)}\cos\phi,$$
$$y = c\sqrt{(1+\nu^2)}\sqrt{(1-\mu^2)}\sin\phi,$$
$$z = c\nu\mu.$$

The line-element ds for this system of coordinates is given by

$$ds^2 = \frac{c^2(\nu^2+\mu^2)}{1-\mu^2}d\mu^2 + c^2(1+\nu^2)(1-\mu^2)\,d\phi^2 + \frac{c^2(\nu^2+\mu^2)}{1+\nu^2}d\nu^2;$$

whence, if we identify μ, ϕ, ν with α, β, γ respectively, we have

$$\frac{1}{h_1} = \frac{c\sqrt{(\nu^2+\mu^2)}}{\sqrt{(1-\mu^2)}}, \qquad \frac{1}{h_2} = c\sqrt{(1+\nu^2)}\sqrt{(1-\mu^2)}, \qquad \frac{1}{h_3} = \frac{c\sqrt{(\nu^2+\mu^2)}}{\sqrt{(1+\nu^2)}},$$

and, supposing that $\nu = \nu_0$ is the equation to the free surface, the equation (11) becomes

$$\frac{\partial\zeta}{\partial t} = -\frac{1}{c\sqrt{(\mu^2+\nu_0^2)}}\frac{\partial}{\partial\mu}\left\{\sqrt{(1-\mu^2)}\,h\,\mathrm{U}\right\} - \frac{1}{c\sqrt{(\nu_0^2+1)}}\frac{\partial}{\partial\phi}\left\{\frac{h\,\mathrm{V}}{\sqrt{(1-\mu^2)}}\right\}.$$

We have already neglected on the right small terms of the order h compared with those retained; we now propose to make the further hypothesis that the spheroidal surface of the ocean is of small ellipticity ϵ. In this case c will be small and ν_0 large, in such a manner however that $c\nu_0$ is finite and equal to the polar radius a; further $1/\nu_0^2$ will be approximately equal to 2ϵ. Hence we find

$$\frac{\partial\zeta}{\partial t} = -\frac{1}{a}\left[\frac{\partial}{\partial\mu}\left\{\sqrt{(1-\mu^2)}\,h\,\mathrm{U}\right\} + \frac{\partial}{\partial\phi}\left\{\frac{h\,\mathrm{V}}{\sqrt{(1-\mu^2)}}\right\}\right] \quad . \quad . \quad . \quad (12),$$

where the terms omitted on the right are of order h and of order ϵ compared with those retained.

§ 4. *Transformation of the Dynamical Equations.*

Let θ denote the inclination to the axis of z of the normal to the surface $\nu =$ const. through any point; then the direction-cosines of this normal will be

$$\sin\theta\cos\phi, \qquad \sin\theta\sin\phi, \qquad \cos\theta;$$

and the direction-cosines of the normals to the surfaces $\mu = \text{const}$, $\phi = \text{const}$, will be

$$\begin{array}{ccc} -\cos\theta\cos\phi, & -\cos\theta\sin\phi, & \sin\theta, \\ -\sin\phi, & \cos\phi, & 0, \end{array}$$

respectively. Hence we have

$$\begin{aligned} \text{U} &= -(u\cos\phi + v\sin\phi)\cos\theta + w\sin\theta, \\ \text{V} &= v\cos\phi - u\sin\phi, \\ \text{W} &= (u\cos\phi + v\sin\phi)\sin\theta + w\cos\theta, \end{aligned}$$

from which we obtain

$$-\text{U}\cos\theta + \text{W}\sin\theta = u\cos\phi + v\sin\phi.$$

Again

$$h_1\frac{\partial\psi}{\partial\mu} = -\frac{\partial\psi}{\partial x}\cos\phi\cos\theta - \frac{\partial\psi}{\partial y}\sin\phi\cos\theta + \frac{\partial\psi}{\partial z}\sin\theta,$$

and therefore from (2), we find

$$\begin{aligned} h_1\frac{\partial\psi}{\partial\mu} &= -\left(\frac{\partial u}{\partial t} - 2\omega v\right)\cos\phi\cos\theta - \left(\frac{\partial v}{\partial t} + 2\omega u\right)\sin\phi\cos\theta + \frac{\partial w}{\partial t}\sin\theta \\ &= \frac{\partial \text{U}}{\partial t} + 2\omega\text{V}\cos\theta. \end{aligned}$$

Similarly

$$\begin{aligned} h_2\frac{\partial\psi}{\partial\phi} &= \frac{\partial\psi}{\partial y}\cos\phi - \frac{\partial\psi}{\partial x}\sin\phi \\ &= \left(\frac{\partial v}{\partial t} + 2\omega u\right)\cos\phi - \left(\frac{\partial u}{\partial t} - 2\omega v\right)\sin\phi \\ &= \frac{\partial \text{V}}{\partial t} + 2\omega(\text{W}\sin\theta - \text{U}\cos\theta), \end{aligned}$$

and

$$\begin{aligned} h_3\frac{\partial\psi}{\partial\nu} &= \frac{\partial\psi}{\partial x}\sin\theta\cos\phi + \frac{\partial\psi}{\partial y}\sin\theta\sin\phi + \frac{\partial\psi}{\partial z}\cos\theta \\ &= \left(\frac{\partial u}{\partial t} - 2\omega v\right)\sin\theta\cos\phi + \left(\frac{\partial v}{\partial t} + 2\omega u\right)\sin\theta\sin\phi + \frac{\partial w}{\partial t}\cos\theta \\ &= \frac{\partial \text{W}}{\partial t} - 2\omega\text{V}\sin\theta. \end{aligned}$$

Allowing for the differences in the notation, the three equations just obtained agree with those given by Professor LAMB.* If we suppose U, V, W, ψ, each proportional to $e^{i\lambda t}$, they may be written

* 'Hydrodynamics,' p. 344.

$$\left.\begin{aligned} i\lambda \mathrm{U} + 2\omega \mathrm{V} \cos\theta &= h_1 \frac{\partial \psi}{\partial \mu}, \\ i\lambda \mathrm{V} - 2\omega (\mathrm{U} \cos\theta - \mathrm{W} \sin\theta) &= h_2 \frac{\partial \psi}{\partial \phi}, \\ i\lambda \mathrm{W} - 2\omega \mathrm{V} \sin\theta &= h_3 \frac{\partial \psi}{\partial \nu}. \end{aligned}\right\} \quad \ldots \ldots \quad (13).$$

The equations in the form we have just written them will hold good whatever be the depth of the ocean or the ellipticity of its surface. We now proceed to introduce approximations similar to those of the last section.

In the first place we have, as in § 2, $\overline{\mathrm{W}} = \frac{\partial \zeta}{\partial t}$, and this by (12) we see is of the order h/a compared with $\overline{\mathrm{U}}$ or $\overline{\mathrm{V}}$. Hence, omitting terms of the order h/a, and of order ϵ, compared with those retained, the equations (13) take at the surface the approximate forms

$$\left.\begin{aligned} i\lambda \overline{\mathrm{U}} + 2\omega \overline{\mathrm{V}} \cos\theta &= \frac{\sqrt{(1-\mu^2)}}{a} \frac{\partial \overline{\psi}}{\partial \mu}, \\ i\lambda \overline{\mathrm{V}} - 2\omega \overline{\mathrm{U}} \cos\theta &= \frac{1}{a\sqrt{(1-\mu^2)}} \frac{\partial \overline{\psi}}{\partial \phi}, \\ - 2\omega \overline{\mathrm{V}} \sin\theta &= \frac{\partial \psi}{\partial n'}, \end{aligned}\right\} \quad \ldots \ldots \quad (14)$$

where, in conformity with the notation of § 2, we have denoted by $\partial\psi/\partial n'$ the rate of variation of ψ in the direction of the normal to the surface of the ocean.

From the equations (14) it appears that $\partial\psi/\partial n'$ is a quantity of the same order of magnitude as $\overline{\psi}/a$; also if we apply the operator $h_3 \frac{\partial}{\partial \nu}$ to each of the equations (13), we shall obtain equations which enable us to express $\partial\mathrm{U}/\partial n'$, $\partial\mathrm{V}/\partial n'$, $\partial\mathrm{W}/\partial n'$ in terms of $\overline{\mathrm{U}}$, $\overline{\mathrm{V}}$, $\overline{\mathrm{W}}$, and the surface-values of ψ and its differential coefficients. A little consideration will show that in general $\partial\mathrm{U}/\partial n'$, $\partial\mathrm{V}/\partial n'$, $\partial\mathrm{W}/\partial n'$ must be of the order $\overline{\mathrm{U}}/a$, $\overline{\mathrm{V}}/a$.

Now the approximations introduced in the last section will hold good provided that we may neglect

$$(\gamma_0 - \gamma_1)\left[\frac{\partial}{\partial \gamma} (\mathrm{U}/h_2 h_3)\right] \text{ and } (\gamma_0 - \gamma_1)\left[\frac{\partial}{\partial \gamma} (\mathrm{V}/h_3 h_1)\right]$$

in comparison with $[\mathrm{U}/h_2 h_3]$, $[\mathrm{V}/h_3 h_1]$; or, in our present notation, that we may neglect

$$h\frac{\partial}{\partial n'}\{c^2\sqrt{(\nu^2+\mu^2)}\sqrt{(1-\mu^2)}\,\mathrm{U}\} \text{ and } h\frac{\partial}{\partial n'}\left\{\frac{c^2(\nu^2+\mu^2)\,\mathrm{V}}{\sqrt{(1-\mu^2)}\sqrt{(1+\nu^2)}}\right\}$$

in comparison with

$$\overline{\mathrm{U}}c^2\sqrt{(\nu_0{}^2+\mu^2)}\sqrt{(1-\mu^2)},\quad \overline{\mathrm{V}}\,\frac{c^2(\nu_0{}^2+\mu^2)}{\sqrt{(1-\mu^2)}\sqrt{(1+\nu_0{}^2)}}.$$

But we have seen that $\partial\mathrm{U}/\partial n'$, $\partial\mathrm{V}/\partial n'$ are of the order $\overline{\mathrm{U}}/a$, while

$$\frac{\partial}{\partial n'}\{c^2\sqrt{(\nu^2+\mu^2)}\},\quad \frac{\partial}{\partial n'}\left\{\frac{c^2(\nu^2+\mu^2)}{\sqrt{(1+\nu^2)}}\right\}$$

are of the order $c^2\nu_0/a$.

Hence

$$h\frac{\partial}{\partial n'}\{c^2\sqrt{(\nu^2+\mu^2)}\sqrt{(1-\mu^2)}\,\mathrm{U}\},\qquad h\frac{\partial}{\partial n'}\left\{\frac{c^2(\nu^2+\mu^2)\,\mathrm{V}}{\sqrt{(1-\mu^2)}\sqrt{(1+\nu^2)}}\right\}$$

are both of the order

$$\overline{\mathrm{U}}\,\frac{h}{a}\,c^2\nu_0.$$

The approximations will therefore be admissible, provided h/a is a small quantity, that is, provided that the depth of the ocean is small in comparison with the radius of the solid earth, a hypothesis as to the validity of which there can be no doubt.

Returning now to equations (14), and solving for $\overline{\mathrm{U}}$, $\overline{\mathrm{V}}$ we find

$$\overline{\mathrm{U}}\,(\lambda^2-4\omega^2\cos^2\theta)=-i\lambda\,\frac{\sqrt{(1-\mu^2)}}{a}\,\frac{\partial\overline{\psi}}{\partial\mu}+\frac{2\omega\cos\theta}{a\sqrt{(1-\mu^2)}}\,\frac{\partial\overline{\psi}}{\partial\phi},$$

$$\overline{\mathrm{V}}\,(\lambda^2-4\omega^2\cos^2\theta)=-2\omega\cos\theta\,\frac{\sqrt{(1-\mu^2)}}{a}\,\frac{\partial\overline{\psi}}{\partial\mu}-\frac{i\lambda}{a\sqrt{(1-\mu^2)}}\,\frac{\partial\overline{\psi}}{\partial\phi}.$$

But we have rigorously

$$\cos\theta=\mu\,\frac{\sqrt{(\nu_0{}^2+1)}}{\sqrt{(\nu_0{}^2+\mu^2)}},$$

and therefore, with errors of the order of the ellipticity, we may replace $\cos\theta$ by μ. Hence, finally, we obtain the values of $\overline{\mathrm{U}}$, $\overline{\mathrm{V}}$ with errors of the order h/a, ϵ compared with their true values in the form

$$\left.\begin{aligned}\overline{\mathrm{U}}&=-\frac{i\lambda\sqrt{(1-\mu^2)}}{a(\lambda^2-4\omega^2\mu^2)}\,\frac{\partial\overline{\psi}}{\partial\mu}+\frac{2\omega\mu}{a\sqrt{(1-\mu^2)}\,(\lambda^2-4\omega^2\mu^2}\,\frac{\partial\overline{\psi}}{\partial\phi},\\ \overline{\mathrm{V}}&=-\frac{2\omega\mu\sqrt{(1-\mu^2)}}{a(\lambda^2-4\omega^2\mu^2)}\,\frac{\partial\overline{\psi}}{\partial\mu}-\frac{i\lambda}{a\sqrt{(1-\mu^2)}\,(\lambda^2-4\omega^2\mu^2)}\,\frac{\partial\overline{\psi}}{\partial\phi}.\end{aligned}\right\}\quad . \quad . \quad (15).$$

Substituting these values in the right-hand member of (12), we obtain

$$i\lambda a^2\zeta = \frac{\partial}{\partial\mu}\left\{\frac{i\lambda h(1-\mu^2)}{\lambda^2-4\omega^2\mu^2}\frac{\partial\bar{\psi}}{\partial\mu} - \frac{2\omega\mu h}{\lambda^2-4\omega^2\mu^2}\frac{\partial\bar{\psi}}{\partial\phi}\right\}$$
$$+\frac{\partial}{\partial\phi}\left\{\frac{2\omega\mu h}{\lambda^2-4\omega^2\mu^2}\frac{\partial\bar{\psi}}{\partial\mu} + \frac{i\lambda h}{(1-\mu^2)(\lambda^2-4\omega^2\mu^2)}\frac{\partial\bar{\psi}}{\partial\phi}\right\} \quad . \quad . \quad . \quad (16).$$

Hence, provided that λ be not equal to zero, we have

$$a^2\zeta = \frac{\partial}{\partial\mu}\left\{\frac{h(1-\mu^2)}{\lambda^2-4\omega^2\mu^2}\frac{\partial\bar{\psi}}{\partial\mu} - \frac{2\omega\mu h}{i\lambda(\lambda^2-4\omega^2\mu^2)}\frac{\partial\bar{\psi}}{\partial\phi}\right\}$$
$$+\frac{\partial}{\partial\phi}\left\{\frac{2\omega\mu h}{i\lambda(\lambda^2-4\omega^2\mu^2)}\frac{\partial\bar{\psi}}{\partial\mu} + \frac{h}{(1-\mu^2)(\lambda^2-4\omega^2\mu^2)}\frac{\partial\bar{\psi}}{\partial\phi}\right\} \quad . \quad . \quad (17).$$

This equation, in conjunction with the pressure equation (9) of § 2, serves to determine ψ, ζ in terms of μ, ϕ. It is equivalent to the well-known equation used by LAPLACE in the 'Mécanique Céleste.'* Omitting from consideration for the present the types of motion defined by $\lambda = 0$, we propose in the present paper to discuss only those solutions which are symmetrical with respect to the axis of rotation. In order that such solutions may exist, we must suppose that h, ψ are independent of ϕ; the equation (17) will then reduce to

$$\frac{\partial}{\partial\mu}\left\{\frac{h(1-\mu^2)}{f^2-\mu^2}\frac{\partial\bar{\psi}}{\partial\mu}\right\} = 4a^2\omega^2\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

where for brevity we have put $\lambda/2\omega = f$.

§ 5. *Integration by Means of Zonal Harmonics.*

Suppose that ζ is expressible as a series of zonal harmonics of the form

$$\zeta = \sum_{n=1}^{n=\infty} C_n P_n(\mu).$$

Neglecting the ellipticity of the surface, we may at once write down the value at the surface of the potential due to this distribution; we have, namely,

$$\bar{v}' = \sum_{n=1}^{n=\infty} \frac{4\pi\rho a}{2n+1} C_n P_n(\mu),$$

* Part I., Book IV., § 3.

where the density ρ is expressed in gravitational units. But if σ denote the mean density of the earth as a whole, including the ocean, we have, with the same degree of approximation,

$$g = \tfrac{4}{3}\pi\sigma a,$$

and therefore

$$\bar{v}' = \sum_{n=1}^{n=\infty} \frac{3\rho}{(2n+1)\sigma} g C_n P_n(\mu).$$

As this only involves the ratio ρ/σ, it is independent of the unit of mass employed.

Next suppose that the surface-value of the disturbing potential can be expressed by means of the series

$$\Sigma\gamma_n P_n(\mu).$$

Then equation (9) of § 2 gives

$$\begin{aligned}\bar{\psi} &= \bar{v}' - g\zeta + \bar{v} \\ &= -\Sigma\left[g\left(1 - \frac{3\rho}{(2n+1)\sigma}\right)C_n - \gamma_n\right]P_n(\mu),\end{aligned}$$

or, if we write for brevity,

$$\left.\begin{aligned} g_n &= g\left(1 - \frac{3\rho}{(2n+1)\sigma}\right), \\ \Gamma_n &= \gamma_n - g_n C_n \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (19),$$

we have

$$\bar{\psi} = \Sigma\Gamma_n P_n(\mu) \quad \ldots \ldots \ldots \quad (20).$$

Now, from the equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_n}{d\mu}\right\} + n(n+1)P_n = 0$$

which defines the zonal harmonics, we find

$$(1-\mu^2)\frac{dP_n}{d\mu} = -n(n+1)\int^{\mu} P_n\, d\mu.$$

Hence, if we integrate (18) with respect to μ, we obtain

$$\begin{aligned}\frac{h(1-\mu^2)}{f^2-\mu^2}\frac{\partial\bar{\psi}}{\partial\mu} &= 4a^2\omega^2\Sigma C_n\int^{\mu} P_n\, d\mu \\ &= A - 4a^2\omega^2(1-\mu^2)\Sigma\frac{C_n}{n(n+1)}\frac{dP_n}{d\mu},\end{aligned}$$

where A is an arbitrary constant, which may be seen to be zero by putting $\mu = \pm 1$. Therefore

$$h\,(1-\mu^2)\frac{\partial\bar{\psi}}{\partial\mu} = -\,4a^2\omega^2\,(1-\mu^2)\,\Sigma\,\frac{C_n}{n\,(n+1)}\,(f^2-\mu^2)\,\frac{dP_n}{d\mu} \quad . \quad . \quad (21).$$

But, by well known properties of the zonal harmonics, we have

$$\begin{aligned}(1-\mu^2)\frac{dP_n}{d\mu} &= -\,n\,(n+1)\int^{\mu} P_n\,d\mu \\ &= -\,\frac{n\,(n+1)}{2n+1}\int^{\mu}\left(\frac{dP_{n+1}}{d\mu}-\frac{dP_{n-1}}{d\mu}\right)d\mu \\ &= -\,\frac{n\,(n+1)}{2n+1}\,(P_{n+1}-P_{n-1})\end{aligned}$$

no arbitrary constant being necessary since both sides vanish when $\mu = 1$; and therefore

$$\begin{aligned}(f^2-\mu^2)\frac{dP_n}{d\mu} &= (f^2-1)\frac{dP_n}{d\mu} - \frac{n\,(n+1)}{2n+1}\,(P_{n+1}-P_{n-1}) \\ &= (f^2-1)\frac{dP_n}{d\mu} - \frac{n\,(n+1)}{2n+1}\left\{\frac{1}{2n+3}\left(\frac{dP_{n+2}}{d\mu}-\frac{dP_n}{d\mu}\right) - \frac{1}{2n-1}\left(\frac{dP_n}{d\mu}-\frac{dP_{n-2}}{d\mu}\right)\right\},\end{aligned}$$

whence

$$\begin{aligned}(f^2-\mu^2)\frac{dP_n}{d\mu} = &-\,\frac{n\,(n+1)}{(2n+1)\,(2n+3)}\,\frac{dP_{n+2}}{d\mu} \\ &+\left(f^2-1+\frac{2n\,(n+1)}{(2n-1)\,(2n+3)}\right)\frac{dP_n}{d\mu} - \frac{n\,(n+1)}{(2n-1)\,(2n+1)}\,\frac{dP_{n-2}}{d\mu}.\end{aligned}$$

This relation will hold good when $n = 1$, provided we replace $dP_{-1}/d\mu$ by zero.

Thus the right-hand member of (21) is equal to

$$4a^2\omega^2\,(1-\mu^2)\sum_{=1}^{n=\infty}\left[\frac{C_{n-2}}{(2n-3)\,(2n-1)} - C_n\left(\frac{f^2-1}{n\,(n+1)}+\frac{2}{(2n-1)\,(2n+3)}\right)+\frac{C_{n+2}}{(2n+3)\,(2n+5)}\right]\frac{dP_n}{d\mu}.$$

The left-hand member may in virtue of (20) be written in the form

$$h\,(1-\mu^2)\,\Sigma\Gamma_n\,\frac{dP}{d\mu}.$$

Equating the two members and dividing by $4\omega^2a^2\,(1-\mu^2)$, we obtain

$$\begin{aligned}&\frac{h}{4\omega^2a^2}\,\Sigma\Gamma_n\,\frac{dP_n}{d\mu} \\ &\quad = \Sigma\,\frac{dP_n}{d\mu}\left\{\frac{C_{n-2}}{(2n-3)\,(2n-1)} - C_n\left(\frac{f^2-1}{n\,(n+1)}+\frac{2}{(2n-1)\,(2n+3)}\right)+\frac{C_{n+2}}{(2n+3)\,(2n+5)}\right\}.\end{aligned}$$

Hence, provided h be constant, the two members will be identical, if for all positive integral values of n we have

$$\frac{C_{n-2}}{(2n-3)(2n-1)} - C_n\left(\frac{f^2-1}{n(n+1)} + \frac{2}{(2n-1)(2n-3)}\right) + \frac{C_{n+2}}{(2n+3)(2n+5)} = \frac{h\Gamma_n}{4\omega^2 a^2} \quad (22),$$

it being understood that $C_0 = 0$, $C_{-1} = 0$.

This is on the hypothesis that the depth is constant; a more general hypothesis would be to suppose that h is of the form

$$k + l(1-\mu^2)$$

where k and l are constants.

Assuming this form for h, the left-hand member of (21) becomes

$$k(1-\mu^2)\Sigma\Gamma_n\frac{dP_n}{d\mu} + l(1-\mu^2)\Sigma\Gamma_n(1-\mu^2)\frac{dP_n}{d\mu},$$

which, by the properties proved above, is equal to

$$k(1-\mu^2)\Sigma\Gamma_n\frac{dP_n}{d\mu}$$
$$-\,l(1-\mu^2)\,\Sigma\left[\frac{(n-2)(n-1)}{(2n-3)(2n-1)}\Gamma_{n-2} - \frac{2n(n+1)}{(2n-1)(2n+3)}\Gamma_n + \frac{(n+2)(n+3)}{(2n+3)(2n+5)}\Gamma_{n+2}\right]\frac{dP_n}{d\mu}.$$

Identifying this with the right-hand member, we obtain

$$\frac{C_{n-2}}{(2n-3)(2n-1)} - C_n\left(\frac{f^2-1}{n(n+1)} + \frac{2}{(2n-1)(2n+3)}\right) + \frac{C_{n+2}}{(2n+3)(2n+5)}$$
$$= \frac{k\Gamma_n}{4\omega^2a^2} - \frac{l}{4\omega^2a^2}\left\{\frac{(n-2)(n-1)}{(2n-3)(2n-1)}\Gamma_{n-2} - \frac{2n(n+1)}{(2n-1)(2n+3)}\Gamma_n + \frac{(n+2)(n+3)}{(2n+3)(2n+5)}\Gamma_{n+2}\right\} \quad (22\text{A}).$$

On introducing the values of Γ_n from (19), equations (22), (22A) may be written

$$\frac{C_{n-2}}{(2n-3)(2n-1)} - C_n\left(\frac{f^2-1}{n(n+1)} + \frac{2}{(2n-1)(2n+3)} - \frac{hg_n}{4\omega^2a^2}\right)$$
$$+ \frac{C_{n+2}}{(2n+3)(2n+5)} = \frac{h\gamma_n}{4\omega^2a^2} \quad \ldots\ldots\ldots \quad (23),$$

$$C_{n-2}\left\{\frac{1-(n-1)(n-2)\,lg_{n-2}/4\omega^2a^2}{(2n-3)(2n-1)}\right\}$$
$$- C_n\left\{\frac{(f^2-1)}{n(n+1)} + \frac{2\{1-n(n+1)\,lg_n/4\omega^2a^2\}}{(2n-1)(2n+3)} - \frac{kg_n}{4\omega^2a^2}\right\}$$
$$+ C_{n+2}\left\{\frac{1-(n+2)(n+3)\,lg_{n+2}/4\omega^2a^2}{(2n+3)(2n+5)}\right\}$$
$$= \frac{k\gamma_n}{4\omega^2a^2} - \frac{l}{4\omega^2a^2}\left\{\frac{(n-2)(n-1)}{(2n-3)(2n-1)}\gamma_{n-2} - \frac{2n(n+1)}{(2n-1)(2n+3)}\gamma_n + \frac{(n+2)(n+3)}{(2n+3)(2n+5)}\gamma_{n+2}\right\}. \quad (23\text{A})$$

The law of variable depth which we have assumed appears to be the most general law which will lead to a difference-relation connecting the successive C's of order not higher than the second. We shall confine ourselves chiefly to the case where the depth is uniform, but the following remark with respect to the more general case seems worthy of attention.

If we put for brevity

$$\xi_n = \frac{1 - n(n+1)\,lg_n/4\omega^2a^2}{(2n+1)(2n+3)}, \quad \eta_{n-2} = \frac{1 - n(n+1)\,lg_n/4\omega^2a^2}{(2n-1)(2n+1)},$$

$$\mathrm{L}_n = \frac{f^2-1}{n(n+1)} + \frac{2\{1 - n(n+1)\,lg_n/4\omega^2a^2\}}{(2n-1)(2n+3)} - \frac{kg_n}{4\omega^2a^2},$$

and suppose all the γ's zero, so that there is no disturbing force, equations (23A) may be written

$$\left.\begin{array}{ll} -\mathrm{L}_1\mathrm{C}_1 + \eta_1\mathrm{C}_3 = 0, & -\mathrm{L}_2\mathrm{C}_2 + \eta_2\mathrm{C}_4 = 0, \\ \xi_1\mathrm{C}_1 - \mathrm{L}_3\mathrm{C}_3 + \eta_3\mathrm{C}_5 = 0, & \xi_2\mathrm{C}_2 - \mathrm{L}_4\mathrm{C}_4 + \eta_4\mathrm{C}_6 = 0, \\ \xi_3\mathrm{C}_3 - \mathrm{L}_5\mathrm{C}_5 + \eta_5\mathrm{C}_7 = 0, & \xi_4\mathrm{C}_4 - \mathrm{L}_6\mathrm{C}_6 + \eta_6\mathrm{C}_8 = 0, \\ \cdots\cdots & \cdots\cdots \end{array}\right\} \quad . \quad . \quad . \quad (24).$$

Now, suppose l in the expression $k + l(1-\mu^2)$ for the depth is of the form $\dfrac{4\omega^2a^2}{r(r+1)g_r}$, where r is an integer, and for greater definiteness let us suppose that r is even.

Then $\xi_r = 0$ and $\eta_{r-2} = 0$, and therefore the equations (24) will all be satisfied if

$$\begin{array}{r} -\mathrm{L}_2\mathrm{C}_2 + \eta_2\mathrm{C}_4 = 0, \\ \xi_2\mathrm{C}_2 - \mathrm{L}_4\mathrm{C}_4 + \eta_4\mathrm{C}_6 = 0, \\ \cdots\cdots \\ \xi_{r-4}\mathrm{C}_{r-4} - \mathrm{L}_{r-2}\mathrm{C}_{r-2} \qquad\qquad = 0, \\ \xi_{r-2}\mathrm{C}_{r-2} - \mathrm{L}_r\mathrm{C}_r + \eta_r\mathrm{C}_{r+2} \qquad = 0, \\ -\mathrm{L}_{r+2}\mathrm{C}_{r+2} + \eta_{r+2}\mathrm{C}_{r+4} = 0, \\ \xi_{r+2}\mathrm{C}_{r+2} - \mathrm{L}_{r+4}\mathrm{C}_{r+4} + \eta_{r+4}\mathrm{C}_{r+6} = 0, \\ \cdots\cdots \end{array}$$

and all the C's with odd suffixes vanish.

Further, these will be satisfied if C_{r+2}, C_{r+4}, . . . are all zero, provided λ is a root of the equation

$$\begin{vmatrix} -L_2, & \eta_2, & 0, & 0, & \cdots & & \\ \xi_2, & -L_4, & \eta_4, & 0, & \cdots & & \\ 0, & \xi_4, & -L_6, & \eta_6 & \cdots & & \\ \cdots & & & & & & \\ \cdots & & & & -L_{r-6}, & \eta_{r-6}, & 0 \\ \cdots & & & & \xi_{r-6}, & -L_{r-4}, & \eta_{r-4} \\ \cdots & & & & 0, & \xi_{r-4}, & -L_{r-2} \end{vmatrix} = 0.$$

It follows that there exist certain types of free oscillation for which all the values of C with suffixes greater than r are zero. For these types the height of the surface-waves will be expressible by a finite series of terms terminating with a term involving P_r.

In like manner, if the disturbing force be derivable from a potential function of the form of a second order harmonic, the equations which determine the forced oscillations are

$$-L_2C_2 + \eta_2C_4 = \frac{k\gamma_2}{4\omega^2a^2} + \frac{4}{7}\frac{l\gamma_2}{4\omega^2a^2},$$

$$\xi_2C_2 - L_4C_4 + \eta_4C_6 = -\frac{2\,.\,3}{5\,.\,7}\frac{l\gamma_2}{4\omega^2a^2},$$

$$\xi_4C_4 - L_6C_6 + \eta_6C_8 = 0,$$

$$\cdots\cdots$$

$$\xi_{r-4}C_{r-4} - L_{r-2}C_{r-2} = 0,$$

$$\xi_{r-2}C_{r-2} - L_rC_r + \eta_rC_{r+2} = 0,$$

$$-L_{r+2}C_{r+2} + \eta_{r+2}C_{r+4} = 0.$$

$$\cdots\cdots$$

If we suppose C_{r+2}, C_{r+4}, . . . all zero, the first $r/2$ of these equations will serve to determine C_2, C_4, . . . C_r in terms of γ_2, while the remaining equations will be satisfied identically. Thus the forced tides for the law of depth in question will be expressible by a finite series of terms terminating with a term involving P_r.

This general law does not hold when $r = 2$, owing to the presence of a term in the right-hand member of the second equation.

The fact that for these laws of depth the tide-heights could be expressed by finite series, instead of by the infinite series usually required, was originally proved by LAPLACE in the 'Mécanique Céleste.'*

* Part I., Book IV., § 5.

§ 6. *The Period-Equation for the Free Oscillations.*

Returning to the case of uniform depth, we see that the equations (23) divide themselves into two groups, in one of which only even suffixes and in the other of which only odd suffixes are involved. We therefore conclude that the types of oscillation divide themselves into two classes, in the former of which the height of the surface-waves will be expressible entirely by harmonics of even order, and in the latter by harmonics of odd order alone. An exactly similar treatment is applicable to each of these classes; we shall therefore select for discussion the former set, contenting ourselves as regards the latter with merely stating results.

Denote by L_n the expression

$$\frac{f^2 - 1}{n(n+1)} + \frac{2}{(2n-1)(2n+3)} - \frac{hg_n}{4\omega^2 a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Then, putting all the γ's equal to zero, the types of free oscillation will be determined by the equations

$$\left.\begin{aligned}
&-C_2 L_2 + \frac{C_4}{7.9} = 0 \\
&\frac{C_2}{5.7} - C_4 L_4 + \frac{C_6}{11.13} = 0 \\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
&\frac{C_{n-2}}{(2n-3)(2n-1)} - C_n L_n + \frac{C_{n+2}}{(2n+3)(2n+5)} = 0 \\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (26).$$

At first sight it might appear that whatever be the value of λ these equations will serve to determine $C_4, C_6, \ldots$ in succession in terms of C_2, whereas we know that this should only be possible for certain determinate values of λ corresponding to the different periods of free oscillation. The manner in which these values of λ are to be determined involves arguments similar to those used by KELVIN* in justification of the procedure of LAPLACE with reference to the forced oscillations after it had been attacked by AIRY† and FERREL‡.

From equations (26), we obtain by actual solution

$$-\frac{C_4}{7.9} = -C_2 L_2$$

$$\frac{C_6}{7.9.11.13} = C_2 \begin{vmatrix} -L_2, & \dfrac{1}{7.9} \\ \dfrac{1}{5.7}, & -L_4 \end{vmatrix}$$

* 'Phil. Mag.', 1875. *Cf.* also an analogous problem treated by NIVEN, 'Phil. Trans.,' 1880, Part I., p. 133 *et seq.*

† "Tides and Waves," § 3.

‡ "Tidal Researches" (U. S. Coast Survey 1873).

and in general

$$(-1)^{n/2}\frac{C_{n+2}}{7.9.11\ldots(2n+5)}=C_2\begin{vmatrix} -L_2, & \frac{1}{7.9}, & 0, & 0 & \ldots \\ \frac{1}{5.7}, & -L_4, & \frac{1}{11.13}, & 0 & \ldots \\ 0, & \frac{1}{9.11}, & -L_6, & \frac{1}{15.17} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots & -L_{n-2}, & \frac{1}{(2n-1)(2n+1)} \\ 0, & 0, & \ldots\; 0, & \frac{1}{(2n-3)(2n-1)}, & -L_n \end{vmatrix}$$

or, denoting the determinant which multiplies C_2 in the last equation written down by Δ_n,

$$C_{n+2}=(-1)^{n/2}\,7.9.11\ \ldots\ (2n+5)\,\Delta_n C_2,\quad (n=2, 4, 6, \ldots\ldots)$$

Now the determinant Δ_n is an algebraic polynomial of degree $n/2$ in f^2. If therefore we equate it to zero, we should obtain an algebraic equation of degree $n/2$. If f^2 has as its value any of the $n/2$ roots of this equation the first $n/2$ of equations (26) will be consistent, while C_{n+2} will vanish. By increasing the value of n we shall approximate more and more closely to the case where an infinite number of such equations are satisfied, while we shall impose an additional condition on the C's, viz. :—that at some stage one of them must vanish. Though in general in the actual motion none of the quantities C_2, C_4, ... are zero, they are however subject to an important restriction, namely, that the series C_2, C_4 . . . . must form a converging series, and therefore we must satisfy the equation

$$\underset{n=\infty}{L^t} C_{n+2}=0.$$

The latter equation may be regarded as the period-equation for the free oscillations. It follows that as n is increased the roots of the equation $\Delta_n=0$, which make $C_{n+2}=0$, must approach closer and closer to certain definite limiting values, which correspond to the different periods of free oscillation, and that the series C_2, C_4 . . . . calculated in succession from equations (26) can only form a convergent series when f has one of these values.

Now we see that $\Delta_n=0$ is the equation obtained by eliminating the C's from the set of equations

$$\left.\begin{aligned}
&-C_2L_2+\frac{C_4}{7.9}=0,\\
&\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\frac{C_{r-2}}{(2r-3)(2r-1)}-C_rL_r+\frac{C_{r+2}}{(2r+3)(2r+5)}=0,\\
&\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\frac{C_{n-2}}{(2n-3)(2n-1)}-C_nL_n\qquad\qquad\qquad\quad=0
\end{aligned}\right\}\quad .\quad .\quad .\quad .\quad . \quad (27).$$

From these we obtain

$$\frac{C_{r-2}/C_r}{(2r-3)(2r-1)}=L_r-\frac{\dfrac{1}{(2r+3)(2r+5)}}{C_r/C_{r+2}},$$

$$=L_r-\frac{\dfrac{1}{(2r+1)(2r+3)^2(2r+5)}}{\dfrac{C_r/C_{r+2}}{(2r+1)(2r+3)}},$$

and therefore by successive applications

$$C_{r-2}/C_r=(2r-3)(2r-1)\left[L_r-\frac{\dfrac{1}{(2r+1)(2r+3)^2(2r+5)}}{L_{r+2}}-\ldots-\frac{\dfrac{1}{(2n-3)(2n-1)^2(2n+1)}}{L_n}\right].$$

Thus the eliminant of equations (27) can be expressed in the form

$$L_2=\frac{\dfrac{1}{7.9}}{C_2/C_4}=\frac{\dfrac{1}{5.7^2.9}}{L_4}-\frac{\dfrac{1}{9.11^2.13}}{L_6}-\ldots-\frac{\dfrac{1}{(2n-3)(2n-1)^2(2n+1)}}{L_n}.$$

We therefore see that the roots of the equation $\Delta_n=0$ are the roots of

$$L_2-\frac{\dfrac{1}{5.7^2.9}}{L_4}-\frac{\dfrac{1}{9.11^2.13}}{L_6}-\ldots-\frac{\dfrac{1}{(2n-3)(2n-1)^2(2n+1)}}{L_n}=0.$$

This form for the equation $\Delta_n=0$ has an advantage over the determinantal form, in that it enables us at once to proceed to the limit when n is made infinitely great, and thus to express the period-equation for the free oscillations by means of the transcendental equation

$$L_2-\frac{\dfrac{1}{5.7^2.9}}{L_4}-\frac{\dfrac{1}{9.11^2.13}}{L_6}-\ldots\ ad\ inf.=0\quad .\quad .\quad .\quad .\quad .\quad .\quad (28).$$

We may however obtain a number of alternative forms for our period-equation. From (27) we find

$$\frac{C_{r+2}/C_r}{(2r+3)(2r+5)} = L_r - \frac{\dfrac{1}{(2r-3)(2r-1)^2(2r+1)}}{\dfrac{C_r/C_{r-2}}{(2r-1)(2r+1)}},$$

and therefore by successive applications

$$\frac{C_{r+2}/C_r}{(2r+3)(2r+5)} = L_r - \frac{\dfrac{1}{(2r-3)(2r-1)^2(2r+1)}}{L_{r-2}} - \ldots - \frac{\dfrac{1}{5.7^2.9}}{L_2}.$$

Thus, we have

$$L_r = \frac{\dfrac{1}{(2r-3)(2r-1)^2(2r+1)}}{\dfrac{C_r/C_{r-2}}{(2r-1)(2r+1)}} + \frac{\dfrac{1}{(2r+1)(2r+3)^2(2r+5)}}{\dfrac{C_r/C_{r+2}}{(2r+1)(2r+3)}}$$

$$= \frac{\dfrac{1}{(2r-3)(2r-1)^2(2r+1)}}{L_{r-4}} - \frac{\dfrac{1}{(2r-7)(2r-5)^2(2r-3)}}{L_{r-4}} - \ldots - \frac{\dfrac{1}{5.7^2.9}}{L_2}$$

$$+ \frac{\dfrac{1}{(2r+1)(2r+3)^2(2r+5)}}{L_{r+2}} - \frac{\dfrac{1}{(2r+5)(2r+7)^2(2r+9)}}{L_{r+4}} - \ldots - \frac{\dfrac{1}{(2n-3)(2n-7)^2(2n+1)}}{L_n}.$$

This is an alternative form for the equation $\Delta_n = 0$; by making n infinite, we obtain as an alternative form of the period-equation

$$L_r - \frac{\dfrac{1}{(2r-3)(2r-1)^2(2r+1)}}{L_{r-2}} - \frac{\dfrac{1}{(2r-7)(2r-5)^2(2r-3)}}{L_{r-4}} - \ldots - \frac{\dfrac{1}{5.7^2.9}}{L_2}$$

$$- \frac{\dfrac{1}{(2r+1)(2r+3)^2(2r+5)}}{L_{r+2}} - \frac{\dfrac{1}{(2r+5)(2r+7)^2(2r+9)}}{L_{r+4}} - \ldots \textit{ad inf.} = 0 \quad (29),$$

where r is any even integer.

§ 7. *Numerical Solution of the Period-Equation.*

The method I have used to solve the above equation will perhaps be best explained by giving a numerical example in detail.

Taking for the ratio of the mean density of the earth to that of water the value given by BOYS,* we deduce

* 'Roy. Soc. Proc.,' vol. 56, 1894, p. 132.

$$\rho/\sigma = \cdot 18093,$$

whence

$$g_2/g = \cdot 89144 \qquad g_{12}/g = \cdot 97829$$
$$g_4/g = \cdot 93969 \qquad g_{14}/g = \cdot 98128$$
$$g_6/g = \cdot 95825 \qquad g_{16}/g = \cdot 98355$$
$$g_8/g = \cdot 96807 \qquad g_{18}/g = \cdot 98533$$
$$g_{10}/g = \cdot 97415 \qquad g_{20}/g = \cdot 98676$$

the values of g_n/g approximating closer and closer to unity as n increases.

Next take $hg/4\omega^2a^2 = \frac{1}{40}$, which corresponds to a depth of $\frac{1}{2880}$ of the earth's radius, or about 7260 feet. With this value of the depth we find

$$\mathrm{L}_2 = \frac{1}{2\,.\,3}\left(\frac{\lambda^2}{4\omega^2} - 1\right) + \cdot 07295, \qquad \mathrm{L}_{12} = \frac{1}{12\,.\,13}\left(\frac{\lambda^2}{4\omega^2} - 1\right) - \cdot 021237,$$
$$\mathrm{L}_4 = \frac{1}{4\,.\,5}\left(\frac{\lambda^2}{4\omega^2} - 1\right) + \cdot 002482, \qquad \mathrm{L}_{14} = \frac{1}{14\,.\,15}\left(\frac{\lambda^2}{4\omega^2} - 1\right) - \cdot 022143,$$
$$\mathrm{L}_6 = \frac{1}{6\,.\,7}\left(\frac{\lambda^2}{4\omega^2} - 1\right) - \cdot 011835, \qquad \mathrm{L}_{16} = \frac{1}{16\,.\,17}\left(\frac{\lambda^2}{4\omega^2} - 1\right) - \cdot 022746,$$
$$\mathrm{L}_8 = \frac{1}{8\,.\,9}\left(\frac{\lambda^2}{4\omega^2} - 1\right) - \cdot 017184, \qquad \mathrm{L}_{18} = \frac{1}{18\,.\,19}\left(\frac{\lambda^2}{4\omega^2} - 1\right) - \cdot 023168,$$
$$\mathrm{L}_{10} = \frac{1}{10\,.\,11}\left(\frac{\lambda^2}{4\omega^2} - 1\right) - \cdot 019777, \qquad \mathrm{L}_{20} = \frac{1}{20\,.\,21}\left(\frac{\lambda^2}{4\omega^2} - 1\right) - \cdot 023476.$$

Introduce for brevity the notation

$$\mathrm{H}_n = \frac{\frac{1}{(2n+1)(2n+3)^2(2n+5)}}{\mathrm{L}_n} - \frac{\frac{1}{(2n-3)(2n-1)^2(2n+1)}}{\mathrm{L}_{n-2}} - \ldots - \frac{\frac{1}{5\,.\,7^2\,.\,9}}{\mathrm{L}_2},$$

$$\mathrm{K}_n = \frac{\frac{1}{(2n-3)(2n-1)^2(2n+1)}}{\mathrm{L}_n} - \frac{\frac{1}{(2n+1)(2n+3)^2(2n+5)}}{\mathrm{L}_{n+2}} - \ldots - \textit{ad inf.}$$

The period-equation (29) may then be written

$$\mathrm{L}_n - \mathrm{H}_{n-2} - \mathrm{K}_{n+2} = 0 \quad \ldots \ldots \quad (30).$$

Suppose now that $\lambda^2/4\omega^2$ has a value found by equating to zero one of the quantities L_n, say for example L_8; putting $\mathrm{L}_8 = 0$, we obtain with the numerical values given above

$$\frac{\lambda^2}{4\omega^2} = 2\cdot 23726,$$

and with this value for $\lambda^2/4\omega^2$ we find

$$L_2 = \cdot 27916, \qquad L_{10} = - \cdot 008529,$$
$$L_4 = \cdot 064345, \qquad L_{12} = - \cdot 013306,$$
$$L_6 = \cdot 017624, \qquad L_{14} = - \cdot 016251.$$

The value of $\log \frac{1}{(2n+1)(2n+3)^2(2n+5)}$ is

$$\text{for } n = 2,\ \bar{4}\cdot 6566, \qquad \text{for } n = 8,\ \bar{6}\cdot 8898,$$
$$,,\ n = 4,\ \bar{5}\cdot 8490, \qquad ,,\ n = 10,\ \bar{6}\cdot 5564,$$
$$,,\ n = 6,\ \bar{5}\cdot 3034, \qquad ,,\ n = 12,\ \bar{6}\cdot 2769.$$

With these values we find for the successive convergents to the continued fraction H_6

$$\cdot 001141, \quad \cdot 001217, \quad \cdot 001219,$$

while the successive convergents to the continued fraction K_{10} are

$$- \cdot 000910, \quad - \cdot 000939, \quad - \cdot 000940, \ldots$$

It will be observed that these continued fractions converge with great rapidity; so long as the depth of the ocean is not less than that we are here using, I find that when $\lambda^2/4\omega^2$ has a value in the neighbourhood of a root of the equation $L_n = 0$, the continued fractions H_{n-2}, K_{n+2} are represented without sensible error by their fourth convergents, while in many cases the second convergents will form a sufficiently accurate approximation to their values; this rapid convergence of course greatly facilitates the numerical computation. In practice, the simplest method of evaluating the continued fractions is to assume that, for a sufficiently large value of n, $K_n = 0$, and then to compute K_{n-2}, K_{n-4}, &c., in succession from the formula

$$K_{n-2} = \frac{\frac{1}{(2n-7)(2n-5)^2(2n-3)}}{L_{n-2} - K_n}.$$

Thus, in the present instance, we may put $K_{16} = 0$, and deduce

$$\log K_{14} = n\bar{4}\cdot 06, \quad \log K_{12} = n\bar{4}\cdot 436, \quad \log K_{10} = n\bar{4}\cdot 9730,$$

whence, as above,

$$K_{10} = - \cdot 000940.$$

In like manner, in order to evaluate H_{n-2}, we assume that, for some sufficiently small value of r, $H_r = 0$, and then compute H_{r+2}, H_{r+4}, . . . , H_{n-2} in turn by means of the formula

$$H_{r+2} = \frac{\frac{1}{(2r+5)(2r+7)^2(2r+9)}}{L_{r+2} - H_r}.$$

In the present case we find

$$\log H_2 = \bar{3}\cdot 2107, \quad \log H_4 = \bar{3}\cdot 0516, \quad \log H_6 = \bar{3}\cdot 0860,$$

whence

$$H_6 = \cdot 001219,$$

and

$$L_8 - H_6 - K_{10} = - \cdot 001219 + \cdot 000940$$
$$= - \cdot 000279.$$

This being a small quantity we conclude that there is a root of the period-equation differing but slightly from the value assumed for $\lambda^2/4\omega^2$, namely, 2·23726. A closer approximation will be found by using this value in H_6, K_{10}, and again equating $L_8 - H_6 - K_{10}$ to zero; in other words, by putting

$$L_8 = + \cdot 000279.$$

The second approximation to the root is therefore given by

$$\lambda^2/4\omega^2 = 2\cdot 25735.$$

Taking this value, and proceeding as before, we find

$$L_8 - H_6 - K_{10} = \cdot 000279 - \cdot 001183 + \cdot 000961$$
$$= \cdot 000057.$$

We have now found that, when $\lambda^2/4\omega^2 = 2\cdot 23726$,

$$L_8 - H_6 - K_{10} = - \cdot 000279,$$

and when $\lambda^2/4\omega^2 = 2\cdot 25735$,

$$L_8 - H_6 - K_{10} = + \cdot 000057,$$

whence, by interpolation, we conclude that

$$L_8 - H_6 - K_{10} = 0,$$

when

$$\lambda^2/4\omega^2 = 2\cdot 25394.$$

In general we shall at this stage obtain a sufficiently close approximation to the root sought, as may be verified by actual substitution. Should however great accuracy be desired, we may re-start the computation, using the value already found as a first approximation, and so continue until the desired degree of accuracy is attained. The number of cases in which I have found a repetition of the process necessary is however extremely limited.

By the method here sketched I have calculated the first six roots of the period-equation for four different depths of the ocean corresponding to the values $\frac{1}{40}$, $\frac{1}{20}$, $\frac{1}{10}$, $\frac{1}{5}$, respectively, for $hg/4\omega^2a^2$. These depths are equivalent to about 7260, 14520, 29040, 58080 feet respectively, and the results are embodied in the following tables :—

TABLE I.

	Approximate value of $\lambda^2/4\omega^2$, computed from equation $L_n = 0$.	Corrected value of $\lambda^2/4\omega^2$.	Period of oscillation expressed in sidereal time.	Corresponding period when there is no rotation.
			h. m.	h. m.
Depth 7,260 feet ($hg/4\omega^2a^2 = \frac{1}{40}$); $\rho/\sigma =$ ·18093.				
$n = 2$	·56230	·44155	18 3·5	32 49
$n = 4$	·95036	·96357	12 13·5	17 30
$n = 6$	1·4971	1·5224	9 43·5	11 58
$n = 8$	2·2373	2·2539	7 59·6	9 5
$n = 10$	3·1755	3·1867	6 43·3	7 20
$n = 12$	4·3130	4·3209	5 46·4	6 9
Depth 14,520 feet ($hg/4\omega^2a^2 = \frac{1}{20}$); $\rho/\sigma =$ ·18093.				
$n = 2$	·69600	·62473	15 11·0	23 12
$n = 4$	1·4202	1·4368	10 0·7	12 23
$n = 6$	2·5032	2·5168	7 33·8	8 28
$n = 8$	3·9798	3·9882	6 0·5	6 26
$n = 10$	5·8544	5·8600	4 57·4	5 11
$n = 12$	8·1283	8·1322	4 12·5	4 21
Depth 29,040 feet ($hg/4\omega^2a^2 = \frac{1}{10}$); $\rho/\sigma =$ ·18093.				
$n = 2$	·96344	·92506	12 28·6	16 25
$n = 4$	2·3599	2·3707	7 47·6	8 45
$n = 6$	4·5155	4·5224	5 38·6	5 59
$n = 8$	7·4649	7·4691	4 23·5	4 33
$n = 10$	11·2123	11·2150	3 35·0	3 40
$n = 12$	15·7588	15·7609	3 1·4	3 4
Depth 58,080 feet ($hg/4\omega^2a^2 = \frac{1}{5}$); $\rho/\sigma =$ ·18093.				
$n = 2$	1·4983	1·4785	9 52·1	11 35
$n = 4$	4·2393	4·2453	5 49·4	6 11
$n = 6$	8·5402	8·5437	4 6·3	4 14
$n = 8$	14·4352	14·4371	3 9·5	3 13
$n = 10$	21·9275	21·9293	2 33·8	2 36
$n = 12$	31·0202	31·0212	2 9·3	2 10

By a comparison of the 2nd and 3rd columns it will be noticed that in most cases the roots of the frequency-equation are given at once with a fair degree of accuracy by simply solving the equations $L_n = 0$, and that this approximation improves the greater n becomes. In the fifth column I have given the periods of oscillation for an ocean of the same depth when the rotation is annulled, calculated by means of the formula

$$\frac{\lambda^2}{4\omega^2} = \frac{n(n+1)hg_n}{4\omega^2 a^2},$$

where ω now denotes a constant such that $\pi/\omega = 12$ hours. It will be seen that the approximation obtained by omitting the rotation continually improves with increasing values of n, but in no case will it lead to as accurate a result as the formula

$$\frac{\lambda^2}{4\omega^2} - 1 = n(n+1)\left\{\frac{hg_n}{4\omega^2 a^2} - \frac{2}{(2n-1)(2n+3)}\right\}.$$

For instance, taking the case $hg/4\omega^2 a^2 = \frac{1}{40}$, $n = 8$, the error introduced by using the first formula amounts to about 14 per cent., whereas the second form gives the frequency with an error less than one per cent. of its true value.

§ 8. *Unsymmetrical Types.*

An exactly similar method of treatment is applicable to the types which are represented by a series of harmonics of odd order; the period-equation for these types is given by

$$L_n - H_{n-2} - K_{n+2} = 0$$

where n now denotes an odd integer and H_n, K_n denote respectively the continued fractions

$$\frac{\dfrac{1}{(2n+1)(2n+3)^2(2n+5)}}{L_n} - \frac{\dfrac{1}{(2n-3)(2n-1)^2(2n+1)}}{L_{n-2}} - \ldots - \frac{\dfrac{1}{3\,.\,5^2\,.\,7}}{L_1},$$

$$\frac{\dfrac{1}{(2n-3)(2n-1)^2(2n+1)}}{L_n} - \frac{\dfrac{1}{(2n+1)(2n+3)^2(2n+5)}}{L_{n+2}} - \ldots \textit{ad inf.}$$

Treating this case in the same manner as the last, I have found the first six roots and the corresponding periods of oscillation for the four depths employed as follows:—

TABLE II.

	Approximate value of $\lambda^2/4\omega^2$.	Corrected value of $\lambda^2/4\omega^2$.	Period of oscillation. h. m.	Corresponding period without rotation. h. m.
Depth, 7,260 feet $(hg/4\omega^2a^2 = \frac{1}{40})$; $\rho/\sigma = \cdot 18093$.				
$n = 1$	·24095	·15491	30 29·3	59 17
$n = 3$	·74340	·70890	14 15·2	22 49
$n = 5$	1·2002	1·2270	10 50·0	14 13
$n = 7$	1·8426	1·8633	8 47·5	10 20
$n = 9$	2·6815	2·6951	7 18·6	8 7
$n = 11$	3·7193	3·7287	6 12·9	6 41
Depth, 14,520 feet $(hg/4\omega^2a^2 = \frac{1}{20})$; $\rho/\sigma = \cdot 18093$.				
$n = 1$	·28191	·22204	25 28·0	41 55
$n = 3$	1·0201	1·0160	11 54·3	16 8
$n = 5$	1·9132	1·9300	8 38·3	10 3
$n = 7$	3·1919	3·2025	6 42·3	7 18
$n = 9$	4·8672	4·8741	5 26·1	5 44
$n = 11$	6·9414	6·9461	4 33·2	4 44
Depth, 29,040 feet $(hg/4\omega^2a^2 = \frac{1}{10})$; $\rho/\sigma = \cdot 18093$.				
$n = 1$	·36381	·32658	20 59·9	29 39
$n = 3$	1·57362	1·57822	9 33·1	11 25
$n = 5$	3·3391	3·34816	6 33·5	7 7
$n = 7$	5·8906	5·8959	4 56·5	5 10
$n = 9$	9·2387	9·2421	3 56·8	4 4
$n = 11$	13·3856	13·3880	3 16·8	3 21
Depth, 58,080 feet $(hg/4\omega^2a^2 = \frac{1}{5})$; $\rho/\sigma = \cdot 18093$.				
$n = 1$	·52763	·50650	16 51·7	20 58
$n = 3$	2·6806	2·6853	7 19·4	8 4
$n = 5$	6·1911	6·1957	4 49·3	5 1
$n = 7$	11·2879	11·2906	3 34·3	3 39
$n = 9$	17·9816	17·9833	2 49·8	2 52
$n = 11$	26·2742	26·2753	2 20·5	2 22

§ 9. *Numerical Computation of the Height of the Surface-Waves.*

We have next to evaluate the quantities C_2, C_4 . . . ; when once the periods have been determined this will present no difficulty. Suppose we are dealing with the type whose frequency approximates to the root of the equation $L_n = 0$; we have seen in the preceding sections how to evaluate K_{n+2}, K_{n+4}, . . . and H_{n-2}, H_{n-4}, . . . Also we have

$$C_{r+2}/C_r = (2r+3)(2r+5)\,K_{r+2},$$
$$C_{r-2}/C_r = (2r-3)(2r-1)\,H_{r-2},$$

and therefore

$$C_{n+2} = (2n+3)(2n+5)\,K_n\,C_n,$$
$$C_{n+4} = (2n+3)(2n+5)(2n+7)(2n+9)\,K_{n+2}\,K_{n+4}\,C_n,$$
$$\cdots\cdots\cdots$$
$$C_{n-2} = (2n-1)(2n-3)\,H_{n-2}\,C_n,$$
$$C_{n-4} = (2n-1)(2n-3)(2n-5)(2n-7)\,H_{n-2}\,H_{n-4}\,C_n.$$
$$\cdots\cdots\cdots$$

Thus the height of the surface-waves is given by

$$\zeta = C_n e^{i\lambda t}\left[\begin{array}{l} \ldots + (2n-1)(2n-3)(2n-5)(2n-7)\,H_{n-2}\,H_{n-4}\,P_{n-4} \\ + (2n-1)(2n-3)\,H_{n-2}\,P_{n-2} + P_n + (2n+3)(2n+5)\,K_{n+2}\,P_{n+2} \\ + (2n+3)(2n+5)(2n+7)(2n+9)\,K_{n+2}\,K_{n+4}\,P_{n+4} + \ldots \end{array}\right]$$

where λ is the root of the frequency-equation in question, and C_n is an arbitrary constant.

Continuing with the particular numerical example dealt with in § 7, we take

$$\frac{\lambda^2}{4\omega^2} = 2\cdot 25394, \qquad \text{or} \qquad \frac{\lambda}{2\omega} = 1\cdot 5014$$

and deduce

$$L_2 = \cdot 281944, \qquad L_{12} = -\cdot 013199,$$
$$L_4 = \cdot 065179, \qquad L_{14} = -\cdot 016171,$$
$$L_6 = \cdot 018021, \qquad L_{16} = -\cdot 01814,$$
$$L_8 = \cdot 000232, \qquad L_{18} = -\cdot 01951,$$
$$L_{10} = -\cdot 008378.$$

Neglecting K_{20}, we find

$$\log K_{18} = n\,\bar{5}\cdot 535, \qquad \log K_{16} = n\,\bar{5}\cdot 7785, \qquad \log K_{14} = n\,\bar{4}\cdot 0698,$$
$$\log K_{12} = n\,\bar{4}\cdot 4397, \qquad \log K_{10} = n\,\bar{4}\cdot 9812,$$

and in like manner

$$\log H_2 = \bar{3}\cdot 2064, \qquad \log H_4 = \bar{3}\cdot 0458, \qquad \log H_6 = \bar{3}\cdot 0753\,;$$

from these we deduce

$$\log (C_2/C_4) = \bar{2}\cdot7505, \qquad \log (C_{10}/C_8) = n\,\bar{1}\cdot5822,$$

$$\log (C_4/C_6) = \bar{1}\cdot0414, \qquad \log (C_{12}/C_{10}) = n\,\bar{1}\cdot1994,$$

$$\log (C_6/C_8) = \bar{1}\cdot3653, \qquad \log (C_{14}/C_{12}) = n\,\bar{2}\cdot9636,$$

$$\log (C_{16}/C_{14}) = n\,\bar{2}\cdot7885,$$

$$\log (C_{18}/C_{16}) = n\,\bar{2}\cdot647\,;$$

whence, finally

$$C_2/C_8 = \cdot0014, \qquad C_{10}/C_8 = -\cdot3821,$$

$$C_4/C_8 = \cdot0255, \qquad C_{12}/C_8 = +\cdot0605,$$

$$C_6/C_8 = \cdot2319, \qquad C_{14}/C_8 = -\cdot0056,$$

$$C_{16}/C_8 = +\cdot0003,$$

$$C_{18}/C_8 = -\cdot00002.$$

Combining with our solution a second, obtained by changing the sign of i wherever it occurs, we obtain a solution in the real form

$$\zeta = C_8 \cos(\lambda t + \epsilon) \begin{bmatrix} \cdot0014P_2 + \cdot0255P_4 + \cdot2319P_6 + P_8 \\ - \cdot3821P_{10} + \cdot0605P_{12} - \cdot0056P_{14} + \cdot0003P_{16} - \ldots \end{bmatrix},$$

where C_8, ϵ are arbitrary constants.

This determines the type of oscillation for that particular mode which is in question. It will be seen that the coefficient of P_8 predominates, and that consequently the deformation of the surface will be similar in character to that which takes place when there is no rotation, in which case the height of the surface-waves is expressed by a single harmonic term. The nodal circles will however be displaced from their positions when the rotation is annulled.

§ 10. *Numerical Expressions for the Height of the Surface-Waves.*

By the method illustrated in the preceding section I have computed the series which indicate the types of oscillation for each of the forty-eight cases for which the periods are tabulated in §§ 7, 8; these series are given in the following tables. To obtain the height of the surface-waves, the series here given must be multiplied by a simple harmonic function of the time of arbitrary amplitude and phase, but whose period is found from the corresponding entry in the preceding tables.

TABLE III.—Heights of Surface-Waves for Symmetrical Types.

$hg/4\omega^2a^2 = \frac{1}{40}$; (7,260 feet).

$n = 2$	$P_2 - 1\cdot2678P_4 + \cdot5267P_6 - \cdot1097P_8 + \cdot0137P_{10} - \cdot0012P_{12} + \cdot0001P_{14} - \ldots$
$n = 4$	$\cdot2373P_2 + P_4 - \cdot8753P_6 + \cdot2595P_8 - \cdot0403P_{10} + \cdot0039P_{12} - \cdot0003P_{14} + \ldots$
$n = 6$	$\cdot0269P_2 + \cdot2714P_4 + P_6 - \cdot5453P_8 + \cdot1139P_{10} - \cdot0132P_{12} + \cdot0010P_{14} - \cdot0001P_{16} + \ldots$
$n = 8$	$\cdot0014P_2 + \cdot0255P_4 + \cdot2319P_6 + P_8 - \cdot3821P_{10} + \cdot0605P_{12} - \cdot0056P_{14} + \cdot0003P_{16} - \ldots$
$n = 10$	$\ldots + \cdot0012P_4 + \cdot0196P_6 + \cdot1977P_8 + P_{10} - \cdot2934P_{12} + \cdot0373P_{14} - \cdot0028P_{16} + \cdot0001P_{18} - \ldots$
$n = 12$	$\ldots + \cdot0009P_6 + \cdot0150P_8 + \cdot1715P_{10} + P_{12} - \cdot2380P_{14} + \cdot0252P_{16} - \cdot0016P_{18} + \cdot0001P_{20} - \ldots$

$hg/4\omega^2a^2 = \frac{1}{20}$; (14,520 feet).

$n = 2$	$P_2 - \cdot7484P_4 + \cdot1707P_6 - \cdot0188P_8 + \cdot0012P_{10} - \cdot0001P_{12} + \ldots$
$n = 4$	$\cdot1286P_2 + P_4 - \cdot4070P_6 + \cdot0594P_8 - \cdot0046P_{10} + \cdot0002P_{12} - \ldots$
$n = 6$	$\cdot0069P_2 + \cdot1311P_4 + P_6 - \cdot2556P_8 + \cdot0262P_{10} - \cdot0015P_{12} + \cdot0001P_{14} - \ldots$
$n = 8$	$\cdot0002P_2 + \cdot0062P_4 + \cdot1127P_6 + P_8 - \cdot1840P_{10} + \cdot0144P_{12} - \cdot0007P_{14} + \ldots$
$n = 10$	$\ldots + \cdot0002P_4 + \cdot0048P_6 + \cdot0969P_8 + P_{10} - \cdot1432P_{12} + \cdot0090P_{14} - \cdot0003P_{16} + \ldots$
$n = 12$	$\ldots + \cdot0001P_6 + \cdot0037P_8 + \cdot0845P_{10} + P_{12} - \cdot1170P_{14} + \cdot0062P_{16} - 0002P_{18} + \ldots$

$hg/4\omega^2a^2 = \frac{1}{10}$; (29,040 feet).

$n = 2$	$P_2 - \cdot4029P_4 + \cdot0477P_6 - \cdot0027P_8 + \cdot0001P_{10} - \ldots$
$n = 4$	$\cdot0677P_2 + P_4 - \cdot1989P_6 + \cdot0144P_8 - \cdot0006P_{10} + \ldots$
$n = 6$	$\cdot0017P_2 + \cdot0651P_4 + P_6 - \cdot1259P_8 + \cdot0064P_{10} - \cdot0002P_{12} + \ldots$
$n = 8$	$\ldots + \cdot0015P_4 + \cdot0560P_6 + P_8 - \cdot0912P_{10} + \cdot0036P_{12} - \cdot0001P_{14} + \ldots$
$n = 10$	$\ldots + \cdot0012P_6 + \cdot0482P_8 + P_{10} - \cdot0712P_{12} + \cdot0022P_{14} - \ldots$
$n = 12$	$\ldots + \cdot0009P_8 + \cdot0421P_{10} + P_{12} - \cdot0583P_{14} + \cdot0015P_{16} - \ldots$

$hg/4\omega^2a^2 = \frac{1}{5}$; (58,080 feet).

$n = 2$	$P_2 - \cdot2076P_4 + \cdot0125P_6 - \cdot0004P_8 + \ldots$
$n = 4$	$\cdot0347P_2 + P_4 - \cdot0989P_6 + \cdot0036P_8 - \cdot0001P_{10} + \ldots$
$n = 6$	$\cdot0004P_2 + \cdot0326P_4 + P_6 - \cdot0627P_8 + \cdot0016P_{10} - \ldots$
$n = 8$	$\ldots + \cdot0004P_4 + \cdot0280P_6 + P_8 - \cdot0455P_{10} + 0009P_{12} - \ldots$
$n = 10$	$\ldots + \cdot00$[illegible]$3P_6 + \cdot0241P_8 + P_{10} - \cdot0355P_{12} + \cdot0006P_{14} - \ldots$
$n = 12$	$\ldots + \cdot0002P_8 + \cdot0210P_{10} + P_{12} - \cdot0291P_{14} + \cdot0004P_{16} - \ldots$

TABLE IV.—Heights of Surface-Waves for Unsymmetrical Types.

$hg/4\omega^2a^2 = \frac{1}{40}$; (7,260 feet).

$n = 1$	$P_1 - 1{\cdot}5058P_3 + {\cdot}7106P_5 - {\cdot}1673P_7 + {\cdot}0235P_9 - {\cdot}0022P_{11} + {\cdot}0001P_{13} - \ldots$
$n = 3$	${\cdot}1221P_1 + P_3 - 1{\cdot}0907P_5 + {\cdot}3875P_7 - {\cdot}0701P_9 + {\cdot}0077P_{11} - {\cdot}0006P_{13} + \ldots$
$n = 5$	${\cdot}0161P_1 + {\cdot}2772P_3 + P_5 - {\cdot}6837P_7 + {\cdot}1688P_9 - {\cdot}0226P_{11} + {\cdot}0019P_{13} - {\cdot}0001P_{15} + \ldots$
$n = 7$	${\cdot}0010P_1 + {\cdot}0280P_3 + {\cdot}2521P_5 + P_7 - {\cdot}4498P_9 + {\cdot}0811P_{11} - {\cdot}0083P_{13} + {\cdot}0006P_{15} - \ldots$
$n = 9$	$\ldots + {\cdot}0014P_3 + {\cdot}0224P_5 + {\cdot}2137P_7 + P_9 - {\cdot}3320P_{11} + {\cdot}0468P_{13} - {\cdot}0039P_{15} + {\cdot}0002P_{17} - \ldots$
$n = 11$	$\ldots + {\cdot}0010P_5 + {\cdot}0171P_7 + {\cdot}1837P_9 + P_{11} - {\cdot}2628P_{13} + {\cdot}0304P_{15} - {\cdot}0021P_{17} + {\cdot}0001P_{19} - \ldots$

$hg/4\omega^2a^2 = \frac{1}{20}$; (14,520 feet).

$n = 1$	$P_1 - 1{\cdot}0477P_3 + {\cdot}2986P_5 - {\cdot}0396P_7 + {\cdot}0030P_9 + {\cdot}0001P_{11} + \ldots$
$n = 3$	${\cdot}0778P_1 + P_3 - {\cdot}5478P_5 + {\cdot}0993P_7 - {\cdot}0091P_9 + {\cdot}0005P_{11} - \ldots$
$n = 5$	${\cdot}0048P_1 + {\cdot}1374P_3 + P_5 - {\cdot}3156P_7 + {\cdot}0381P_9 - {\cdot}0025P_{11} + {\cdot}0001P_{13} - \ldots$
$n = 7$	${\cdot}0001P_1 + {\cdot}0068P_3 + {\cdot}1218P_5 + P_7 - {\cdot}2141P_9 + {\cdot}0190P_{11} - {\cdot}0010P_{13} + \ldots$
$n = 9$	$\ldots + {\cdot}0002P_3 + {\cdot}0054P_5 + {\cdot}1043P_7 + P_9 - {\cdot}1611P_{11} + {\cdot}0113P_{13} - {\cdot}0005P_{15} + \ldots$
$n = 11$	$\ldots + {\cdot}0001P_5 + {\cdot}0042P_7 + {\cdot}0903P_9 + P_{11} - {\cdot}1288P_{13} + {\cdot}0074P_{15} - {\cdot}0003P_{17} + \ldots$

$hg/4\omega^2a^2 = \frac{1}{10}$; (29,040 feet).

$n = 1$	$P_1 - {\cdot}6516P_3 + {\cdot}1034P_5 - {\cdot}0073P_7 + {\cdot}0003P_9 - \ldots$
$n = 3$	${\cdot}0471P_1 + P_3 - {\cdot}2726P_5 + {\cdot}0248P_7 - {\cdot}0011P_9 + \ldots$
$n = 5$	${\cdot}0013P_1 + {\cdot}0689P_3 + P_5 - {\cdot}1547P_7 + {\cdot}0093P_9 - {\cdot}0003P_{11} + \ldots$
$n = 7$	$\ldots + {\cdot}0017P_3 + {\cdot}0605P_5 + P_7 - {\cdot}1059P_9 + {\cdot}0047P_{11} - {\cdot}0001P_{13} + \ldots$
$n = 9$	$\ldots + {\cdot}0014P_5 + {\cdot}0519P_7 + P_9 - {\cdot}0800P_{11} + {\cdot}0028P_{13} - {\cdot}0001P_{15} + \ldots$
$n = 11$	$\ldots + {\cdot}0010P_7 + {\cdot}0450P_9 + P_{11} - {\cdot}0641P_{13} + {\cdot}0018P_{15} - \ldots$

$hg/4\omega^2a^2 = \frac{1}{5}$; (58,080 feet).

$n = 1$	$P_1 - {\cdot}3697P_3 + {\cdot}0310P_5 - {\cdot}0011P_7 + \ldots$
$n = 3$	${\cdot}0265P_1 + P_3 - {\cdot}1361P_5 + {\cdot}0063P_7 - {\cdot}0001P_9 + \ldots$
$n = 5$	${\cdot}0003P_1 + {\cdot}0346P_3 + P_5 - {\cdot}0770P_7 + {\cdot}0023P_9 - \ldots$
$n = 7$	$\ldots + {\cdot}0004P_3 + {\cdot}0302P_5 + P_7 - {\cdot}0528P_9 + {\cdot}0012P_{11} - \ldots$
$n = 9$	$\ldots + {\cdot}0003P_5 + {\cdot}0259P_7 + P_9 - {\cdot}0399P_{11} + {\cdot}0007P_{13} - \ldots$
$n = 11$	$\ldots + {\cdot}0003P_7 + {\cdot}0225P_9 + P_{11} - {\cdot}0320P_{13} + {\cdot}0005P_{15} - \ldots$

§ 11. *Forced Tides.*

Leaving now the problem of the free oscillations, let us return to the equations of § 5, when we retain the γ's. It is obvious in the first place that a disturbing force whose potential at the surface is expressible by surface-harmonics of even order alone, or of odd order alone, will give rise to a forced oscillation of like character. Further we may consider separately the effects of the different terms in the disturbing potential and superpose the results. Suppose for example that the surface-value of the disturbing potential is expressible by the single harmonic term

$$\gamma_n P_n(\mu)\, e^{i\lambda t}$$

where we will suppose n even.

The equations (23) which determine the type may be written

$$-C_2L_2 + C_4/7 \,.\, 9 = 0$$
$$C_2/5 \,.\, 7 - C_4L_4 + C_6/11 \,.\, 13 = 0$$
$$\cdots\cdots\cdots\cdots$$
$$C_{n-2}/(2n-3)(2n-1) - C_nL_n + C_{n+2}/(2n+3)(2n+5) = \gamma_n h/4\omega^2a^2$$
$$C_n/(2n+1)(2n+3) - C_{n+2}L_{n+2} + C_{n+4}/(2n+7)(2n+9) = 0$$
$$\cdots\cdots\cdots\cdots$$

with the condition that $C_\infty = 0$; whence we obtain

$$C_{r-2}/C_r = (2r-3)(2r-1)\, H_{r-2}\ (r < n+1)$$
$$C_{r+2}/C_r = (2r+3)(2r+5)\, K_{r+2}\ (r > n-1)$$

and therefore

$$C_n\,\{H_{n-2} + K_{n+2} - L_n\} = \gamma_n h/4\omega^2a^2,$$

or

$$C_n = \frac{\gamma_n h}{4\omega^2a^2(H_{n-2} + K_{n+2} - L_n)}.$$

Thus the height of the tide is given by

$$\zeta = \frac{\gamma_n h}{4\omega^2a^2(H_{n-2}+K_{n+2}-L_n)}\left[\begin{array}{l}\ldots + (2n-1)(2n-3)(2n-5)(2n-7)\,H_{n-2}H_{n-4}P_{n-4} \\ + (2n-1)(2n-3)H_{n-2}P_{n-2} + P_n + (2n+3)(2n+5)K_{n+2}P_{n+2} \\ + (2n+3)(2n+5)(2n+7)(2n+9)\,K_{n+2}K_{n+4}P_{n+4} + \ldots\end{array}\right]$$

The expressions H, K, L all depend on λ the frequency of the disturbing force. It is obvious from the above that the tides become very large when λ approaches a root of the equation

$$L_n - H_{n-2} - K_{n+2} = 0;$$

and this equation, as we have already seen, is the equation which determines the periods of free oscillation.

It is usual in Tidal Theory to express the height of the forced tides in terms of the height of the corresponding "equilibrium-tides." If we denote the height of the equilibrium-tide arising from the disturbing potential in question by $\mathfrak{C}_n P_n(\mu) e^{i\lambda t}$, we see, on omitting all the terms on the left of (23) which depend on inertia, and replacing C_n by $\mathfrak{C}_n$, that

$$hg_n \mathfrak{C}_n / 4\omega^2 a^2 = \gamma_n h / 4\omega^2 a^2 ;$$

and therefore

$$\gamma_n = g_n \mathfrak{C}_n .$$

If then ζ_0 denote the height of the equilibrium-tide, we have

$$\frac{\zeta}{\zeta_0} = \frac{hg_n/4\omega^2 a^2}{(H_{n-2} + K_{n+2} - L_n) P_n} \left[\begin{array}{l} \ldots + (2n-1)(2n-3)(2n-5)(2n-7) H_{n-2} H_{n-4} P_{n-4} \\ \quad + (2n-1)(2n-3) H_{n-2} P_{n-2} + P_n \\ \quad + (2n+3)(2n+5) K_{n+2} P_{n+2} \\ \quad + (2n+3)(2n+5)(2n+7)(2n+9) K_{n+2} K_{n+4} P_{n+4} \\ \quad + \ldots \end{array} \right]$$

The most important practical application of the above theory is the case where the disturbing potential involves only a single harmonic term of the second order, and the period of the disturbance is long compared with the period of rotation. Thus taking $\lambda^2/4\omega^2 = \cdot 00133$, $hg/4\omega^2 a^2 = 1/40$, which corresponds to the case of lunar-fortnightly tides in an ocean of depth 7260 feet, we find

$$L_2 = -\cdot 09349, \qquad L_8 = -\cdot 03105,$$
$$L_4 = -\cdot 04745, \qquad L_{10} = -\cdot 02886,$$
$$L_6 = -\cdot 03561, \qquad L_{12} = -\cdot 02764.$$

whence, neglecting K_{14}, we obtain in succession

$$\log K_{12} = n\bar{4}\cdot 12, \qquad \log K_{10} = n\bar{4}\cdot 432, \qquad \log K_8 = n\bar{4}\cdot 8151,$$
$$\log K_6 = n\bar{3}\cdot 3055, \qquad \log K_4 = n\bar{3}\cdot 9992.$$

Thus

$$L_2 - K_4 = -\cdot 08351,$$
$$\log (C_4/C_2) = n\bar{1}\cdot 7986, \qquad \log (C_6/C_4) = n\bar{1}\cdot 4608, \qquad \log (C_8/C_6) = n\bar{1}\cdot 2216,$$
$$\log (C_{10}/C_8) = n\bar{1}\cdot 033, \qquad \log (C_{12}/C_{10}) = n\bar{2}\cdot 88,$$

$$C_2/\mathfrak{C}_2 = \frac{\frac{1}{40}\frac{g_n}{g}}{K_4 - L_2} = \cdot 2669,$$

and therefore

$$\frac{\zeta}{\zeta_0/P_2} = \cdot 2669P_2 - \cdot 1678P_4 + \cdot 0485P_6 - \cdot 0081P_8 + \cdot 0009P_{10} - \cdot 0001P_{12} + \ldots$$

In the same manner, when $hg/4\omega^2a^2 = \frac{1}{20}$,

$$\frac{\zeta}{\zeta_0/P_2} = \cdot 4079P_2 - \cdot 1671P_4 + \cdot 0285P_6 - \cdot 0027P_8 + \cdot 0002P_{10} - \ldots;$$

when $hg/4\omega^2a^2 = \frac{1}{10}$,

$$\frac{\zeta}{\zeta_0/P_2} = \cdot 5697P_2 - \cdot 1388P_4 + \cdot 0131P_6 - \cdot 0006P_8 + \ldots;$$

and when $hg/4\omega^2a^2 = \frac{1}{5}$,

$$\frac{\zeta}{\zeta_0/P_2} = \cdot 7208P_2 - \cdot 0973P_4 + \cdot 0048P_6 - \cdot 0001P_8 + \ldots$$

The lunar-fortnightly declinational tides have been evaluated by Professor DARWIN* for depths which correspond with the first and third cases given above, the results being expressed in series proceeding according to ascending powers of the variable μ. If we replace the various powers of μ by their values in terms of the zonal harmonics,† we may deduce the following series from those given by Professor DARWIN; when $hg/4\omega^2a^2 = \frac{1}{40}$, we find

$$\frac{\zeta}{\zeta_0/P_2} = \cdot 2889P_2 - \cdot 1755P_4 + \cdot 0490P_6 - \cdot 0079P_8 + \cdot 0009P_{10} - \ldots,$$

while, when $hg/4\omega^2a^2 = \frac{1}{10}$,

$$\frac{\zeta}{\zeta_0/P_2} = \cdot 5969P_2 - \cdot 1385P_4 + \cdot 0126P_6 - \cdot 0006P_8 + \ldots$$

The difference between these expansions and those we have given above, is to be explained by the fact that we have included in our analysis the effects due to the attraction of the water on itself. I have re-computed the lunar-fortnightly tides, starting with the assumption that $\rho/\sigma = 0$, and obtained practically identical results by the two methods.

We see then that the effect of the gravitational attraction of the water is to diminish the tides, as compared with the equilibrium tides, in the first case by about 8 per cent., and in the second by about 5 per cent.

* 'Encyc. Brit.,' *Art.* "Tides." § 18.

† FERRERS: 'Spherical Harmonics,' p. 27.

§ 12. *Lunar-Fortnightly Tides in an Ocean of Variable Depth.*

A similar method of treatment may be employed when the depth follows the less restricted law, of the form $k + l(1 - \mu^2)$, made use of in § 5. The numerical computation is greatly facilitated in this case when l takes the form $\frac{4\omega^2 a^2}{r(r+1)g_r}$ where r is a small integer, since, as we have seen, the series which express the tide-heights will then rapidly terminate.

For example, taking $r = 4$ so that

$$\frac{lg}{4\omega^2 a^2} = \frac{1}{4\,.\,5 g_4/g} = \frac{1}{4\,.\,5\left\{1 - \frac{\rho}{3\sigma}\right\}},$$

which makes the value of l for the earth about 15,454 feet, the values of C_2, C_4 are given by the equations

$$-L_2 C_2 = \left\{\frac{k}{4\omega^2 a^2} + \frac{4}{7}\frac{l}{4\omega^2 a^2}\right\}\gamma_2 = \left\{\frac{kg_2}{4\omega^2 a^2} + \frac{4}{7}\frac{lg_2}{4\omega^2 a^2}\right\}\mathfrak{C}_2,$$

$$\xi_2 C_2 - L_4 C_4 = -\frac{2\,.\,3}{5\,.\,7}\frac{l\gamma_2}{4\omega^2 a^2} = -\frac{2\,.\,3}{5\,.\,7}\frac{lg_2}{4\omega^2 a^2}\mathfrak{C}_2,$$

whilst all the remaining C's vanish. The notation employed is that introduced at the end of § 5.

Taking $\frac{kg}{4\omega^2 a^2} = \frac{1}{5}$, $\rho/\sigma = \cdot 18093$, we find in the case of the lunar-fortnightly tides

$$L_2 = -\cdot 27660, \quad L_4 = -\cdot 23787,$$
$$\log \xi_2 = \bar{2}\cdot 3105,$$

whence we obtain at once

$$C_2 = -\left(\frac{kg}{4\omega^2 a^2} + \frac{4}{7}\frac{lg}{4\omega^2 a^2}\right)\frac{g_2}{g}\frac{\mathfrak{C}_2}{L_2} = \cdot 7426\,\mathfrak{C}_2,$$

$$C_4 = \frac{2\,.\,3}{5\,.\,7}\frac{lg_2}{4\omega^2 a^2}\frac{\mathfrak{C}_2}{L_4} + \frac{\xi_2}{L_4}C_2 = -\cdot 0980\,\mathfrak{C}_2.$$

Thus, if $\frac{hg}{4\omega^2 a^2} = \frac{1}{5} + \frac{1}{4\,.\,5}\frac{g}{g_4}\sin^2\theta$, where θ denotes the co-latitude, which for a system of the dimensions of the earth makes the law of depth

$$(58080 + 15454\sin^2\theta) \text{ feet},$$

we find for the tide-height the expression

$$\zeta = \mathfrak{C}_2\left[\cdot 7426 P_2 - \cdot 0980 P_4\right].$$

Similarly, when

$$h = \frac{4\omega^2 a^2}{g}\left(\frac{1}{10} + \frac{1}{4.5}\frac{g}{g_4}\sin^2\theta\right) = (29040 + 15454\sin^2\theta),$$

$$\zeta = \mathfrak{C}_2[\cdot 6201 P_2 - \cdot 1446 P_4];$$

when

$$h = \frac{4\omega^2 a^2}{g}\left(\frac{1}{20} + \frac{1}{4.5}\frac{g}{g_4}\sin^2\theta\right) = (14520 + 15454\sin^2\theta),$$

$$\zeta = \mathfrak{C}_2[\cdot 5018 P_2 - \cdot 1897 P_4];$$

and when

$$h = \frac{4\omega^2 a^2}{g}\left(\frac{1}{40} + \frac{1}{4.5}\frac{g}{g_4}\sin^2\theta\right) = (7260 + 15454\sin^2\theta),$$

$$\zeta = \mathfrak{C}_2[\cdot 4095 P_2 - \cdot 2248 P_4].$$

Again, if we take

$$l = \frac{4\omega^2 a^2}{6.7 g_6} = 7218,$$

the series for ζ will terminate with a term involving P_6, and the values of C_2, C_4, C_6 must be computed from the equations

$$-L_2 C_2 + \eta_2 C_4 = \left(\frac{kg}{4\omega^2 a^2} + \frac{4}{7}\frac{lg}{4\omega^2 a^2}\right)\frac{\gamma_2}{g} = \left(\frac{kg_2}{4\omega^2 a^2} + \frac{4}{7}\frac{lg_2}{4\omega^2 a^2}\right)\mathfrak{C}_2,$$

$$\xi_2 C_2 - L_4 C_4 = -\frac{2.3}{5.7}\frac{l\gamma_2}{4\omega^2 a^2} = -\frac{2.3}{5.7}\frac{lg_2}{4\omega^2 a^2}\mathfrak{C}_2,$$

$$\xi_4 C_4 - L_6 C_6 = 0;$$

from which we find, when $\frac{kg}{4\omega^2 a^2} = \frac{1}{5}$,

$$C_2 = \cdot 7316\mathfrak{C}_2, \qquad C_4 = -\cdot 0978\mathfrak{C}_2, \qquad C_6 = \cdot 0024\mathfrak{C}_2.$$

Thus for the law of depth

$$h = \frac{4\omega^2 a^2}{g}\left\{\frac{1}{5} + \frac{1}{6.7}\frac{g}{g_6}\sin^2\theta\right\} = 58080 + 7218\sin^2\theta,$$

$$\zeta = \mathfrak{C}_2[\cdot 7316 P_2 - \cdot 0978 P_4 + \cdot 0024 P_6];$$

in like manner, when

$$h = \frac{4\omega^2 a^2}{g}\left\{\frac{1}{10} + \frac{1}{6.7}\frac{g}{g_6}\sin^2\theta\right\} = 29040 + 7218\sin^2\theta,$$

$$\zeta = \mathfrak{C}_2[\cdot 5954 P_2 - \cdot 1426 P_4 + \cdot 0064 P_6];$$

when

$$h = \frac{4\omega^2 a^2}{g}\left\{\frac{1}{20} + \frac{1}{6.7}\frac{g}{g_6}\sin^2\theta\right\} = 14520 + 7218\sin^2\theta,$$

$$\zeta = \mathfrak{C}_2[\cdot 4576 P_2 - \cdot 1821 P_4 + \cdot 0137 P_6];$$

and when

$$h = \frac{4\omega^2 a^2}{g}\left\{\frac{1}{40} + \frac{1}{6.7}\frac{g}{g_6}\sin^2\theta\right\} = 7260 + 7218\sin^2\theta,$$

$$\zeta = \mathfrak{C}_2[\cdot 3457 P_2 - \cdot 2075 P_4 + \cdot 0234 P_6].$$

For other values of l we must employ a method similar to that of the last section. The general formulæ for the computation of the forced tides due to a disturbing potential of the second order are

$$\left.\begin{aligned} -L_2C_2 + \eta_2 C_4 &= \frac{k\gamma_2}{4\omega^2a^2} + \frac{4}{7}\frac{l\gamma_2}{4\omega^2a^2} = \left(\frac{kg_2}{4\omega^2a^2} + \frac{4}{7}\frac{lg_2}{4\omega^2a^2}\right)\mathfrak{C}_2 \\ \xi_2C_2 - L_4C_4 + \eta_4C_6 &= -\frac{2.3}{5.7}\frac{l\gamma_2}{4\omega^2a^2} = -\frac{2.3}{5.7}\frac{lg_2}{4\omega^2a^2}\mathfrak{C}_2 \\ \xi_4C_4 - L_6C_6 + \eta_6C_8 &= 0 \\ \cdots\cdots & \end{aligned}\right\} \quad . \quad . \quad (31),$$

where

$$\xi_n = \frac{1 - n(n+1)\,lg_n/4\omega^2a^2}{(2n+1)(2n+3)}, \qquad \eta_{n-2} = \frac{1 - n(n+1)\,lg_n/4\omega^2a^2}{(2n-1)(2n+1)},$$

$$L_n = \frac{f^2-1}{n(n+1)} + \frac{2\{1 - n(n+1)\,lg_n/4\omega^2a^2\}}{(2n-1)(2n+3)} - \frac{kg_n}{4\omega^2a^2}.$$

Let us introduce the notation

$$H_n = \frac{\xi_n\eta_n}{L_n} - \frac{\xi_{n-2}\eta_{n-2}}{L_{n-2}} - \ldots - \frac{\xi_2\eta_2}{L_2},$$

$$K_n = \frac{\xi_{n-2}\eta_{n-2}}{L_n} - \frac{\xi_n\eta_n}{L_{n+2}} - \ldots$$

Then it may be shewn, as in the last sections, that for values of n greater than 2,

$$C_{n+2}/C_n = K_{n+2}/\eta_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

and therefore the first two of equations (31) may be written

$$-L_2C_2 + \eta_2C_4 = \left\{\frac{kg_2}{4\omega^2a^2} + \frac{4}{7}\frac{lg_2}{4\omega^2a^2}\right\}\mathfrak{C}_2,$$

$$\xi_2C_2 - (L_4 - K_6)\,C_4 = -\frac{2.3}{5.7}\frac{lg_2}{4\omega^2a^2}\mathfrak{C}_2.$$

On solving these equations we obtain

$$\left.\begin{aligned} C_2 &= -\left\{\frac{kg_2}{4\omega^2a^2} + \frac{4}{7}\frac{lg_2}{4\omega^2a^2}\right\}\frac{\mathfrak{C}_2}{L_2 - K_4} + \frac{2.3}{5.7}\frac{lg_2}{4\omega^2a^2}\frac{H_2}{\xi_2}\frac{\mathfrak{C}_2}{L_4 - H_2 - K_6} \\ C_4 &= -\left\{\frac{kg_2}{4\omega^2a^2} + \frac{4}{7}\frac{lg_2}{4\omega^2a^2}\right\}\frac{K_4}{\eta_2}\frac{\mathfrak{C}_2}{L_2 - K_4} + \frac{2.3}{5.7}\frac{lg_2}{4\omega^2a^2}\frac{\mathfrak{C}_2}{L_4 - H_2 - K_6}\end{aligned}\right\} \quad . \quad (33),$$

after which we can deduce C_6, C_8 . . . in succession by means of (32).

For example, taking

$$\frac{kg}{4\omega^2a^2} = \frac{1}{20}, \quad \frac{lg}{4\omega^2a^2} = \frac{1}{30},$$

which makes the law of depth for the earth

$$(14520 + 9680 \sin^2\theta) \text{ feet},$$

we find for the lunar-fortnightly tide

$$L_2 = -\cdot13276, \quad L_4 = -\cdot08722, \quad L_6 = -\cdot07583,$$
$$L_8 = -\cdot07156, \quad L_{10} = -\cdot06955, \quad L_{12} = -\cdot0685,$$

and

$$\log \xi_2\eta_2 = \bar{4}\cdot1437, \quad \log \xi_4\eta_4 = n\,\bar{6}\cdot9549, \quad \log \xi_6\eta_6 = \bar{6}\cdot9586,$$
$$\log \xi_8\eta_8 = \bar{5}\cdot4217, \quad \log \xi_{10}\eta_{10} = \bar{5}\cdot578,$$

whence, if we suppose $K_{14} = 0$, we obtain in succession, from the formula

$$K_n = \frac{\xi_{n-2}\,\eta_{n-2}}{L_n - K_{n+2}},$$

$$\log K_{12} = n\,\bar{4}\cdot75, \quad \log K_{10} = n\,\bar{4}\cdot583, \quad \log K_8 = n\,\bar{4}\cdot1062,$$
$$\log K_6 = \bar{4}\cdot0758, \quad \log K_4 = n\,\bar{3}\cdot2025.$$

Also

$$\log H_2 = n\,\bar{3}\cdot0206, \quad \log \xi_2 = \bar{2}\cdot3707, \quad \log \eta_2 = \bar{3}\cdot7730;$$

whence from (33) we find

$$C_2 = \cdot4719\mathfrak{C}_2, \quad C_4 = -\cdot1852\mathfrak{C}_2.$$

Again

$$\begin{aligned} \log (C_6/C_4) &= \log (K_6/\eta_4) = n\,\bar{2}\cdot6976, \\ \log (C_8/C_6) &= \log (K_8/\eta_6) = \bar{2}\cdot3910, \\ \log (C_{10}/C_8) &= \log (K_{10}/\eta_8) = \bar{2}\cdot774,\end{aligned}$$

whence finally

$$C_6 = \cdot 0092, \quad C_8 = \cdot 00023, \quad C_{10} = \cdot 00001;$$

and the expression for the tide-height becomes

$$\zeta = \mathfrak{C}_2 [\cdot 4719 P_2 - \cdot 1852 P_4 + \cdot 0092 P_6 + \cdot 00023 P_8 + \cdot 00001 P_{10} + \ldots].$$

As a further illustration, I have computed the series for the tide-height for the case where $h = \frac{4\omega^2 a^2}{g}\left(\frac{1}{20} - \frac{1}{30}\sin^2\theta\right)$, that is, where the depth is 14,520 feet at the poles and shallows to 4840 feet at the equator. The value of ζ in this case is as follows :—

$$\zeta = \mathfrak{C}_2 [\cdot 3082 P_2 - \cdot 1106 P_4 + \cdot 0467 P_6 - \cdot 0158 P_8 + \cdot 0048 P_{10} - \cdot 0014 P_{12} + \cdot 0004 P_{14} - \cdot 0001 P_{16} + \ldots].$$

When l is positive, that is, when the depth at the equator exceeds that at the poles, the series appear to converge more rapidly than when the depth is uniform, but the opposite is the case when the water is deeper at the poles than at the equator.

§ 13. *Forced Oscillations of Infinitely Long Period.**

If we suppose λ so small that we may neglect $\lambda^2/4\omega^2$, we find, on putting $\lambda^2/4\omega^2 = 0$, for the height of the forced tides the following four series in place of those given in § 11 :—

$$\frac{\zeta}{\zeta_0/P_2} = \cdot 2661 P_2 - \cdot 1671 P_4 + \cdot 0482 P_6 - \cdot 0080 P_8 + \cdot 0009 P_{10} - \cdot 0001 P_{12} + \ldots$$

$$\frac{\zeta}{\zeta_0/P_2} = \cdot 4070 P_2 - \cdot 1666 P_4 + \cdot 0284 P_6 - \cdot 0026 P_8 + \cdot 0002 P_{10} - \ldots$$

$$\frac{\zeta}{\zeta_0/P_2} = \cdot 5689 P_2 - \cdot 1385 P_4 + \cdot 0130 P_6 - \cdot 0006 P_8 + \ldots$$

$$\frac{\zeta}{\zeta_0/P_2} = \cdot 7201 P_2 - \cdot 0973 P_4 + \cdot 0048 P_6 - \cdot 0001 P_8 + \ldots$$

The lunar-fortnightly tides therefore differ only very slightly from tides whose period is infinitely long. The difference between these latter and the solar semi-

* Several of the conclusions of the present section have been previously arrived at by Professor LAMB ('Hydrodynamics,' chapter viii.); but, on account of the important light which they throw on the later sections, I have thought it desirable to treat the questions in some detail, even at the risk o repeating what is already well known.

annual tides will be quite inappreciable, and we may take the above series as giving a good representation of the solar long-period tides, unless the effects of friction become important for such tides.

The fact that when the period of the disturbing force is increased without limit the free surface does not tend to approach its equilibrium form appears at first sight to be at variance with the general laws of oscillating systems. The explanation of this apparent anomaly may perhaps be made clear by considering a simple form of "gyrostatic" system which possesses only two degrees of freedom. In the absence of frictional forces, the general equations of motion of such a system may, by a proper choice of coordinates, be expressed in the form

$$\ddot{x} - \omega\dot{y} + n^2 x = \mathrm{X},$$
$$\ddot{y} + \omega\dot{x} + m^2 y = \mathrm{Y}.*$$

Here x, y denote the generalized coordinates of the system. Of the terms on the left, the terms $\ddot{x}$, $\ddot{y}$ are due to inertia, the terms $\omega\dot{y}$, $\omega\dot{x}$ are described by THOMSON and TAIT as "motional" forces, and the terms n^2x, m^2y as "positional" forces; X, Y are the generalized components of the external disturbing force.

If now x, y, X, Y be supposed proportional to $e^{i\lambda t}$, we find from the above equations

$$-\lambda^2 x - \omega i\lambda y + n^2 x = \mathrm{X},$$
$$-\lambda^2 y + \omega i\lambda x + m^2 y = \mathrm{Y},$$

whence we may obtain x, y in terms of X, Y. When the period of vibration is indefinitely prolonged, λ will approach zero as a limit, and the limiting form of the solution will in general be

$$x = \mathrm{X}/n^2, \qquad y = \mathrm{Y}/m^2.$$

This implies that the displacements will in general tend to acquire their equilibrium-values as the period of the disturbing force is lengthened. There will, however, be an exception to this law if one or both of the positional forces n^2x, m^2y vanish.

Let us first examine the nature of the free oscillations in such cases; omitting X, Y, and supposing that $n = 0$ while m remains finite, we have for the determination of the free motions

$$\left.\begin{aligned} -\lambda^2 x - \omega i\lambda y \qquad\quad &= 0 \\ -\lambda^2 y + \omega i\lambda x + m^2 y &= 0 \end{aligned}\right\}$$

or, if we denote by u, v the generalized velocity-components so that $u = \dot{x} = i\lambda x$, $v = \dot{y} = i\lambda y$,

* THOMSON and TAIT, 'Natural Philosophy,' vol. 1, p. 396 (1886 edition).

$$i\lambda (u - \omega y) = 0,$$
$$- \lambda^2 y + \omega u + m^2 y = 0.$$

These equations will be satisfied if $\lambda = 0$, $u = -\frac{m^2 y}{\omega} =$ const. It follows that the system is capable of a small free steady motion relative to the rotating axes, defined by

$$y = \text{const}, \qquad u = - m^2 y/\omega.$$

If both $n^2 x$, $m^2 y$ are zero, the equations for the free motions become

$$- \lambda^2 x - \omega i \lambda y = 0,$$
$$- \lambda^2 y + \omega i \lambda x = 0\ ;$$

both of which are satisfied by supposing that x, y are small arbitrary constants, and therefore $\lambda = 0$.

In the latter case the equilibrium-state defined by $x = 0$, $y = 0$ is not the only condition of relative equilibrium, but any other configuration of the system in the neighbourhood of this one will also form a configuration of relative equilibrium.

Let us now consider the nature of the limiting forms of the forced oscillations when the period is indefinitely prolonged. In the former case we must suppose that the disturbing forces are such that they do no work when the coordinate x is varied, so that $X = 0$, as otherwise the stability of the system will be destroyed; the equations of motion for the forced oscillations then become

$$i\lambda u - \omega i \lambda y = 0,$$
$$- \lambda^2 y + \omega u + m^2 y = Y\ ;$$

whence

$$u = \omega y = \frac{Y\omega}{\omega^2 + (m^2 - \lambda^2)}$$

and in the limit

$$u = \frac{Y\omega}{m^2 + \omega^2},\ y = \frac{Y}{m^2 + \omega^2}.$$

The velocity-component u will therefore always remain of the same order as the disturbing force Y, while the amplitude of vibration of the coordinate x will tend to increase without limit.

In the latter case the equations of disturbed motion may be written

$$i\lambda u - \omega v = X$$
$$i\lambda v + \omega u = Y$$

whence in the limiting case

$$u = \mathrm{Y}/\omega, \quad v = -\mathrm{X}/\omega.$$

Hence both the velocity-components tend to finite limits while the amplitudes of vibration of both coordinates increase without limit.

The essential characteristics of both cases are (i) that one or more of the generalized coordinates does not appear explicitly in the equations of motion but only the corresponding velocity-component, and hence (ii) that $\lambda = 0$ is one root of the frequency-equation for the free modes of vibration, from which it follows that (α) either free steady motions relatively to the rotating system are possible, or (β) that the configuration of relative equilibrium defined by $x = 0$, $y = 0$ is not isolated. The two conditions (α), (β) may both be expressed by stating that the steady motion defined by $x = 0$, $y = 0$ is not the only form of steady motion of which the system is capable.

The two cases are both illustrated by our problem. For if we suppose the waters of the ocean displaced horizontally in such a manner that the form of the surface is unaltered, we shall evidently obtain a new configuration of relative equilibrium, while, as we shall see in the next section, if we suppose that the fluid is in relative motion in such a manner that the fluid particles are moving along parallels of latitude, it is possible by a proper adjustment of the free surface to ensure that such a motion should be permanent.

The coordinates which depend on the horizontal displacements alone are analogous to the coordinate x in the former of the illustrations we have given above, and to the coordinates x, y in the latter. They do not appear explicitly in the equations of motion, but only through the corresponding velocity-components. We conclude, then, that the horizontal velocities will be of the order of the disturbing forces, whereas the horizontal excursions of the fluid particles will tend to increase without limit as the period is prolonged.

By way of explaining how these circumstances may arise physically, let us suppose for the moment that λ is actually zero, and consequently that the disturbing force is constant. In the case of a system oscillating about a position of equilibrium, the introduction of a constant disturbing force will have the effect of slightly changing the configuration about which oscillations corresponding with the free modes of vibration take place. Suppose now that a disturbing force, such as that which gives rise to the long-period tides, tending to increase the surface-ellipticity of the ocean, is suddenly applied to our rotating system when in a configuration of relative equilibrium. It will immediately set up oscillations, the initial motion being such that each particle will tend towards the position in which it would be in equilibrium under the new disturbing influence. The new position of equilibrium is such that in it there will be more water in equatorial regions, and less water in polar regions, than in the old. Thus the initial motion involves a flow of water directed from the poles towards the equator. The water however coming from higher latitudes into lower will reach these lower latitudes with an amount of rotation less than that which is appropriate for these latitudes if the whole were in a state of steady motion as a rigid body. There are no forces acting which tend to modify the angular momentum about the polar axis of an elementary ring of water, which coincides with a parallel of latitude, and consequently currents will be started, in virtue of which each particle of fluid will move along a parallel of latitude from east to west. The effect of the disturbing force is therefore to modify the state of steady motion about which the free oscillations take place from a uniform rotation of the whole system as a rigid body to a state in which there exist horizontal westerly currents. If, as is usual in dealing with forced oscillations, we suppose the free oscillations to be annulled, we see that the "forced oscillation" arising from such a constant disturbance as we have been considering will be of the nature of a steady motion relatively to the rotating earth, consisting of a westerly flow of the whole ocean, the velocity however varying with the latitude.

In the case of a periodic disturbance of very long period, the motion set up at any instant will be of like character, provided that the viscosity of the fluid is not sufficient to sensibly affect the currents in question in the course of a single period. An equilibrium-theory will only be applicable when the rate of dissipation of such motions is so rapid that they practically disappear in a time which is short compared with the period of the disturbing force. Now in an ocean whose depth is equal to the mean depth of the actual ocean, it seems highly improbable that such currents would be appreciably affected by viscosity in the course of a few months. Hence it appears that the present theory in which the effects of viscosity are totally disregarded will almost certainly give a far better representation of the lunar long-period tides than the equilibrium theory, and most probably also of the long-period solar tides.

§ 14. *Free Steady Motions.*

In the last section we have called attention to the fact that free oscillations of infinite period are possible, or that the system with which we are concerned is capable of free steady motions. We proceed in the present section to examine the nature of these steady motions.

Referring back to §§ 2–4, we see that the general equations of motion of the ocean when free from external disturbing influence, and at the same time supposed steady, so that $\lambda = 0$, can be expressed in the form

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$$\left.\begin{aligned}
\frac{\partial\zeta}{\partial t} &= -\frac{1}{a}\left[\frac{\partial}{\partial\mu}\{\sqrt{(1-\mu^2)}\,h\,\mathrm{U}\} + \frac{\partial}{\partial\phi}\left\{\frac{h\,\mathrm{V}}{\sqrt{(1-\mu^2)}}\right\}\right], \\
\mathrm{U} &= -\frac{1}{2\omega a\mu\sqrt{(1-\mu^2)}}\frac{\partial\psi}{\partial\phi}, \\
\mathrm{V} &= \frac{\sqrt{(1-\mu^2)}}{2\omega a\mu}\frac{\partial\psi}{\partial\mu}, \\
\psi &= v' - g\zeta
\end{aligned}\right\} \quad . \quad . \quad (34).$$

where, as there is no longer any ambiguity, we have omitted the bars from the symbols U, V, ψ, v'.

Since $\partial\zeta/\partial t = 0$ when the motion is steady, we see that the first equation is identically satisfied if we suppose h, ψ both independent of ϕ. In this case we may take

$$\zeta = \Sigma\,\mathrm{C}_n\mathrm{P}_n(\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

where the constants C_n are arbitrary, and deduce

$$\psi = -\Sigma\,g_n\mathrm{C}_n\mathrm{P}_n(\mu);$$

whence

$$\mathrm{V} = -\frac{\sqrt{(1-\mu^2)}}{2\omega a\mu}\Sigma\,g_n\mathrm{C}_n\frac{d\mathrm{P}_n}{d\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

The last equation gives the velocity which must be imposed on the particles of water in latitude $\sin^{-1}\mu$ in order that the free surface may be maintained in the form defined by (35) without any external force. We see that it is theoretically possible to maintain an arbitrary surface-form by correctly distributing the longitudinal velocities of the fluid particles. If however the series (35) involves harmonics of odd order the value of V given by (36) becomes infinite at the equator, and to prevent a flow of liquid across the equator it would be necessary to impose an infinite velocity on the particles of water there. Hence, if the water extend either wholly or partially over both hemispheres, the distribution of velocity and the form of free surface must be symmetrical with respect to the equator, at least so far as concerns that part of the ocean which communicates across the equator.

Conversely, any arbitrary initial distribution of longitudinal velocity symmetrical with respect to the equator may be rendered permanent by an appropriate adjustment of the free surface. These results hold good whatsoever be the law of depth, provided it be a function of the latitude alone.

If ψ be not supposed independent of ϕ, we see from the second and third of equations (34) that the velocity of flow across any element ds inclined at an angle χ to the meridian is

$$\begin{aligned} \mathrm{U} \sin \chi - \mathrm{V} \cos \chi, \\ &= -\frac{1}{2\omega a\mu}\left\{\sin\chi\frac{1}{\sqrt{(1-\mu^2)}}\frac{\partial\psi}{\partial\phi} + \cos\chi\sqrt{(1-\mu^2)}\frac{\partial\psi}{\partial\mu}\right\}, \\ &= -\frac{1}{2\omega\mu}\frac{\partial\psi}{\partial s}. \end{aligned}$$

Hence, if $\partial\psi/\partial s = 0$, there will be no flow across the element ds. It follows that the function ψ may be regarded as a stream-function, the paths of the particles of water always coinciding with the lines

$$\psi = \text{const.}$$

But from the equation of continuity we have, on putting $\partial\zeta/\partial t = 0$, and replacing U, V by their values in terms of ψ,

$$0 = \frac{\partial}{\partial\mu}\left\{\frac{h}{\mu}\frac{\partial\psi}{\partial\phi}\right\} - \frac{\partial}{\partial\phi}\left\{\frac{h}{\mu}\frac{\partial\psi}{\partial\mu}\right\},$$

or

$$\frac{\partial}{\partial\mu}\left(\frac{h}{\mu}\right)\frac{\partial\psi}{\partial\phi} - \frac{\partial}{\partial\phi}\left(\frac{h}{\mu}\right)\frac{\partial\psi}{\partial\mu} = 0,$$

the general solution of which is

$$\psi = f(h/\mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

where $f(h/\mu)$ denotes an arbitrary function of h/μ.

It follows that the stream-lines $\psi =$ const coincide in direction with the lines

$$h/\mu = \text{const} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

Thus, if the depth be a function of the latitude alone, the stream-lines must necessarily coincide with the parallels of latitude, and the only forms of steady motion possible are those in which the water has no latitudinal velocity. In the more general case, the stream-lines of the possible steady motions are given by the equation (38), and from this equation they might at once be traced out on a chart if we had a sufficient knowledge of the depth of the ocean in different parts. In particular, whatever be the law of depth, the equator will be one of the free stream-lines corresponding to an infinite value of h/μ, while the shores will also be stream-lines corresponding to zero values of this expression. An infinite number of stream-lines will converge towards those points where the coast-line intersects the equator, and it is only by passing through one of these points that a particle of water could pass from the northern to the southern hemisphere, or *vice versâ*. As however the velocities at these particular points tend to become infinite, the equations which we have used which involve the neglect of the squares of the velocities will not be applicable to the region immediately surrounding the points in question. It seems most probable that the inclusion of the terms involving the squares of velocities would have the effect of diverting the stream-lines, so as to cause them to follow the coast-lines even in the immediate neighbourhood of the equator.

An attempt to trace out the lines $h/\mu =$ const for the North Atlantic Ocean from data obtainable from the Admiralty charts quickly showed that the theory here put forward is inadequate in itself to explain even the more salient features of the circulation in the region in question. Observation however indicates the direction in which we must look for the defects of this theory. The excessively low temperature of the water beneath the surface* in equatorial regions can only be explained by supposing that this water has travelled thither from higher latitudes, whereas we know that the currents at the surface, for the most part, set from the equator towards the poles. We conclude that the under-currents of the actual ocean differ materially from the surface-currents, and in this respect the actual circulation differs from the types of circulation with which we have hitherto been concerned and which are the only possible types of circulation in our ideal ocean in which the density is uniform. It has been urged by some authorities that the variations in the density of the water arising from differences of temperature, salinity, &c., are the sole causes which maintain ocean currents, but in that we have seen that currents could exist even without such variations, it seems to me to be highly improbable that such is the case, though there can be no doubt, in the light of our present analysis, that these variations are largely effective in determining the course which the currents pursue.

If we suppose that the ocean consists of a number of horizontal layers of different densities, but that the density throughout each stratum is uniform, then for each of the strata a function ψ will exist defined by

$$\psi = \mathrm{V}' + \tfrac{1}{2}\omega^2(x^2 + y^2) - \frac{p}{\rho} + \text{const},$$

and the horizontal velocities for any stratum will be connected with the corresponding function by the equations (34). The equation of continuity for any stratum may be formed as in § 3, provided we replace h by the depth of the stratum in question instead of the depth of the whole ocean. The stream-lines for any stratum will therefore still be given by the equation (38), with this modification in the meaning of the symbol h. It follows that the equator will still be one of the free stream-lines, but the motion elsewhere may be totally different from what it would be if the density were the same throughout. It seems probable then that the result we have obtained with reference to the tendency of the currents to set along the equator will still hold good even when the density is variable, and this conclusion is borne out by

* The principal facts at present known in relation to the distribution of temperature in the ocean will be found in the 'Report of the *Challenger* Scientific Results,' 'Chemistry and Physics,' vol. 1.

observation, it being noticeable at a glance at the Admiralty current charts that there is no tendency to cross the equator except in the immediate neighbourhood of the coasts.

The rigorous treatment of the problem of ocean currents, as affected by variations in the density of the water, appears to be hopelessly beyond the powers of mathematical analysis, and I will therefore leave the subject with the brief indications already given in this section, and will conclude the paper with an example illustrating another means by which possibly ocean currents are in part maintained, and which is instructive in showing the very important part played by the rotation of the earth in rendering effective a cause which otherwise could give rise to no sensible currents.

§ 15. *On Currents due to Evaporation and Precipitation.*

A cause which has been advocated* in explanation of ocean currents is the fact that in equatorial regions the amount of water evaporated into the atmosphere largely exceeds that precipitated in the form of rain in these regions. The excess of water in the atmosphere is carried away to be precipitated in temperate and polar regions, thereby giving rise to an excess of precipitation over evaporation in the latter regions. It has been urged with some reason that, as the actual amount of water in equatorial regions does not diminish nor that in polar regions increase from year to year, there must be a continual flow of water from the poles towards the equator. The fact that this flow of water is in the opposite direction to that observed at the surface, which for the most part sets from the equator towards the poles, is explained by attributing the counterflow to undercurrents. If however we subject the question to the test of mathematical analysis, we shall find that though such a flow towards the equator must necessarily exist, it is so slow as to be completely masked by larger currents due to other causes. The flow in question will however give rise indirectly, in consequence of the rotation, to currents which in the absence of dissipative forces would tend to increase without limit. The explanation of this fact will be obvious after the discussions of § 11.

The effects of evaporation and precipitation may be conveniently represented mathematically by an appropriate distribution of sources and sinks over the free surface. This will modify the surface-conditions at the free surface but will not interfere with the dynamical equations. Instead of equating $\overline{\mathrm{W}}$ to $\partial\zeta/\partial t$ we must replace it by a certain function of the position on the surface, independent of the time, but depending on the rate of evaporation and precipitation at the place,

* See an Article by PROCTOR in 'St. Paul's Magazine,' Sept., 1869, reprinted in 'Light Science' (1st series), p. 114.

which over sufficiently long intervals of time we may regard as uniform. A simple law which will serve for purposes of illustration may be chosen as follows :—

$$\overline{\mathrm{W}} = -\alpha \mathrm{P}_2(\mu).$$

The equations with which we have to deal will then be

$$\left.\begin{aligned} \frac{\partial \mathrm{U}}{\partial t} + 2\omega\mu\mathrm{V} &= \frac{\sqrt{(1-\mu^2)}}{a}\frac{\partial\psi}{\partial\mu} \\ \frac{\partial \mathrm{V}}{\partial t} - 2\omega\mu\mathrm{U} &= \frac{1}{a\sqrt{(1-\mu^2)}}\frac{\partial\psi}{\partial\phi} \\ \alpha\mathrm{P}_2(\mu) &= \frac{1}{a}\left[\frac{\partial}{\partial\mu}\{\sqrt{(1-\mu^2)}\,h\mathrm{U}\} + \frac{\partial}{\partial\phi}\left\{\frac{h\mathrm{V}}{\sqrt{(1-\mu^2)}}\right\}\right] \end{aligned}\right\} \quad . \quad . \quad (39),$$

where, if we neglect the attraction due to the surface-inequalities, we may take

$$\psi = -g\zeta.$$

To obtain a particular solution of these equations suppose

$$\psi = -g\zeta = -g(\zeta_0 + \zeta_1 t), \quad \mathrm{U} = \mathrm{U}_0 + \mathrm{U}_1 t, \;\; \mathrm{V} = \mathrm{V}_0 + \mathrm{V}_1 t,$$

where ζ_0, ζ_1, &c., are all independent of t.

Substituting these expressions in the equations (39), and equating coefficients of t, we find

$$2\omega\mu\mathrm{V}_1 = -\frac{g}{a}\sqrt{(1-\mu^2)}\frac{\partial\zeta_1}{\partial\mu}$$

$$2\omega\mu\mathrm{U}_1 = \frac{g}{a}\sqrt{(1-\mu^2)}\frac{\partial\zeta_1}{\partial\phi}$$

$$0 = \frac{1}{a}\left[\frac{\partial}{\partial\mu}\{\sqrt{(1-\mu^2)}\,h\mathrm{U}_1\} + \frac{\partial}{\partial\phi}\left\{\frac{h\mathrm{V}_1}{\sqrt{(1-\mu^2)}}\right\}\right].$$

If h be a function of μ alone, these will be satisfied by

$$\mathrm{U}_1 = 0, \quad \mathrm{V}_1 = -\frac{g\sqrt{(1-\mu^2)}}{2a\omega\mu}\frac{\partial\zeta_1}{\partial\mu},$$

provided ζ_1 be also a function of μ alone.

Next, if we equate the terms independent of t in the two members of (39), we obtain

$$\mathrm{U}_1 + 2\omega\mu\mathrm{V}_0 = -\frac{g}{a}\sqrt{(1-\mu^2)}\frac{\partial\zeta_0}{\partial\mu}$$

$$\mathrm{V}_1 - 2\omega\mu\mathrm{U}_0 = -\frac{g}{a}\frac{1}{\sqrt{(1-\mu^2)}}\frac{\partial\zeta_0}{\partial\phi}$$

$$\alpha\mathrm{P}_2(\mu) = \frac{1}{a}\left[\frac{\partial}{\partial\mu}\{\sqrt{(1-\mu^2)}\,h\mathrm{U}_0\} + \frac{\partial}{\partial\phi}\left\{\frac{h\mathrm{V}_0}{\sqrt{(1-\mu^2)}}\right\}\right].$$

Suppose the system starts from rest in its position of relative equilibrium, so that $\zeta = 0$ when $t = 0$, or $\zeta_0 = 0$. Then

$$U_1 + 2\omega\mu V_0 = 0$$
$$V_1 - 2\omega\mu U_0 = 0,$$

whence,

$$V_0 = -\frac{U_1}{2\omega\mu} = 0$$
$$U_0 = \frac{V_1}{2\omega\mu} = -\frac{g}{4\omega^2 a}\frac{\sqrt{(1-\mu^2)}}{\mu^2}\frac{\partial\zeta_1}{\partial\mu},$$

and therefore

$$\alpha P_2(\mu) = -\frac{g}{4\omega^2 a^2}\frac{\partial}{\partial\mu}\left\{\frac{(1-\mu^2)h}{\mu^2}\frac{\partial\zeta_1}{\partial\mu}\right\}.$$

The last equation gives

$$\frac{(1-\mu^2)h}{\mu^2}\frac{\partial\zeta_1}{\partial\mu} = -\frac{4\omega^2 a^2}{g}\alpha\int P_2(\mu)\,d\mu$$
$$= -\frac{4\omega^2 a^2}{g}\alpha\frac{\mu^3-\mu}{2},$$

no arbitrary constant being added, since both sides vanish when $\mu = \pm 1$. Thus,

$$h\frac{\partial\zeta_1}{\partial\mu} = \frac{2\omega^2 a^2}{g}\alpha\mu^3,$$

or

$$\zeta_1 = \frac{2\omega^2 a^2\alpha}{g}\int\frac{\mu^3}{h}\,d\mu.$$

Suppose for example that h is constant; we shall then obtain

$$\zeta_1 = \tfrac{1}{2}\frac{\omega^2 a^2\alpha}{gh}\mu^4 + \text{const}$$
$$= \tfrac{1}{2}\frac{\omega^2 a^2\alpha}{gh}\{\tfrac{8}{35}P_4 + \tfrac{4}{7}P_2 + \tfrac{1}{5}\} + \text{const.}$$

Choosing the constant so that the mean value of ζ_1 over the surface is zero, we obtain finally

$$\zeta_1 = \tfrac{1}{2}\frac{\omega^2 a^2\alpha}{gh}\{\tfrac{8}{35}P_4 + \tfrac{4}{7}P_2\}.$$

Hence the particular solutions of the differential equations which represent the "forced" motion due to the disturbing influence in question are

$$\zeta = \zeta_1 t = \frac{2\omega^2 a^2 \alpha t}{35gh}\{2P_4 + 5P_2\},$$

$$U = U_0 + U_1 t = -\tfrac{1}{2}\frac{a}{h}\alpha\mu\sqrt{(1-\mu^2)},$$

$$V = V_0 + V_1 t = -\frac{\alpha a \omega t}{h}\mu^2\sqrt{(1-\mu^2)}.$$

We see then that the effects of evaporation and precipitation will be to cause a steady flow of water, not by means of undercurrents only, but by currents sensibly uniform throughout the depth, towards the equator; in addition to these currents the cause in question will give rise to longitudinal currents, not of a steady character, but increasing uniformly with the time, and these will be accompanied by an appropriate continuous deformation of the free surface. Were it not for viscosity these currents would increase without limit and ultimately endanger the stability of the system, but under the action of dissipative forces a steady state must ultimately be attained, in which the rate at which the currents are generated exactly balances that at which they are destroyed. Thus, suppose the type of motion set up is such that if left to itself it would be reduced in the ratio $1 : e$ in a period τ. If U denote the velocity of any particle, the law of variation of U under the influence of viscous forces is then

$$\partial U/\partial t + U/\tau = 0,$$

whereas, if there be no viscosity and the system is subjected to such a disturbance as we have been dealing with, the velocity varies according to the law

$$\partial U/\partial t = f,$$

where f is constant. Equating the rate of increase of the velocity without viscosity to the rate of decrease under the influence of dissipative force, we find that the ultimate state is defined by

$$U/\tau = f, \quad \text{or} \quad U = f\tau.$$

Thus, if the disturbing influence tends to set up one of the possible types of motion of which the system is capable under viscosity, the ultimate velocity of any particle will be that which it would acquire in a period equal to the modulus of decay of the type of motion in question.

By way of numerical illustration, take a year as the unit of time and an inch as the unit of length, and suppose $\alpha = 40$. This will imply an annual rainfall at the poles which exceeds evaporation by 40 inches, and an annual rainfall at the equator which is less than evaporation by 20 inches. Further, suppose $h/a = \frac{1}{2890}$. Then

$$U = -\tfrac{1}{2} \times 2890 \times 40\,\mu\sqrt{(1-\mu^2)}.$$

U will be numerically greatest when $\mu^2 = \frac{1}{2}$, or in latitude 45°, and the greatest value of U is 28,900. The maximum latitudinal velocity will therefore amount to 28,900 inches, or about half-a-mile, per annum. This velocity will be quite inappreciable to observation.

The amount of longitudinal velocity generated in the course of a year in any latitude is given by the formula

$$\frac{\alpha a \omega}{h} \mu^2 \sqrt{(1 - \mu^2)}.$$

This will be greatest when $\mu^2 = \frac{2}{3}$, that is, in latitude 55° nearly, and its greatest value corresponds to a velocity of about $\frac{1}{5}$ of a mile per hour. The maximum current velocity due to this cause would therefore amount to about four miles per hour if the modulus of decay is as long as 20 years.

The most crucial test to which we can subject the theory of ocean currents here put forward will consist in the evaluation of the moduli of decay for the types of motion concerned. This I have endeavoured to do, but as the work involves analytical considerations of a somewhat different character from those which occur in the present work, I have deemed it advisable to present the results in a separate paper.* These results, so far as they are applicable, seem to point to a modulus of decay far in excess of the 20 years here required, but the mathematical difficulties have compelled me, in dealing with friction, to leave the rotation entirely out of account. It appears that in the simpler system so treated, the types of free current motion are far more arbitrary in character than those at which we have arrived by including the rotation. This arises from the fact that our rotating system will be capable of a large number of free oscillatory motions besides those which we have examined in which the periods of oscillation always bear a finite ratio to the period of rotation. As the period of rotation is lengthened, the period of each of these types of oscillation is prolonged, and the possible forms of steady motion where there is no rotation must include the limiting forms of each of these types of oscillatory motion.

The moduli of decay of the free current motions when there is rotation may therefore be very different in value from those obtained in the paper referred to, but it does not seem to me that they could be much less in order of magnitude than the moduli of decay of the principal types of free oscillation. If this prove to be the case, the estimate of 20 years, which we have taken for the modulus of decay, will not be so excessive as might at first sight appear.

Of course, if the water does not cover the whole earth, or if the depth be not uniform, the currents due to the rotation will follow the free stream-lines defined by equation (38) instead of following the parallels of latitude. The rotation will, no doubt, produce its maximum effect when the stream-lines in question coincide with the parallels of latitude, but this circumstance does not alter our main conclusion as to the adequacy of evaporation and other such causes to generate currents quite comparable with those known to exist in the ocean.

* Read before the London Mathematical Society, December 10th, 1896.